A note on truth value in uncertain logic

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\begin{abstract}
Uncertain logic is a generalization of classical logic for dealing with uncertain knowledge via uncertainty theory. The truth value is defined as the uncertain measure that a proposition is true. In this paper, a numerical method for calculating the truth value of uncertain formulas is proposed and some examples are presented.
\end{abstract}

\section{1. Introduction}

Uncertainty theory, founded by Liu (2007) in 2007 to study the behavior of uncertain phenomena, is a branch of mathematics based on nonmonotonicity, monotonocity, self-duality, countable subadditivity, and product measure axioms. It is a new tool to study subjective uncertainty. As an application of uncertainty theory, Liu (2009) proposed a spectrum of uncertain programming which is mathematical programming involving uncertain variables. Liu (2008) introduced an uncertain process as a sequence of uncertain variables indexed by time or space. As a counterpart of Brownian motion, Liu (2009) designed a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments. Following that, uncertain calculus was initiated by Liu (2009) to deal with differentiation and integration of functions of uncertain processes. In addition, Liu (2008) gave the definition of uncertain differential equation and assuming that stock price follows a geometric canonical process, he derived an uncertain stock model and a European option price formula. Furthermore, Peng and Yao (2010) proposed a new uncertain stock model and other option price formulas. Other references related to uncertainty theory are Gao (2009), Gao, Gao, and Ralescu (2010), Qin and Kar (2009), Qin, Kar, and Li (2009), You (2009), Peng and Iwamura (2010), Zhu (2010), etc. For exploring the recent developments of uncertainty theory, the readers may consult Liu (2010).

In classical logic, each proposition is either true or false. However, a proposition containing vague predicates may be neither true nor false. In order to deal with propositions with vague predicates, multi-valued logic was proposed by Lukasiewicz and extended by many researchers such as Adams and Levine (1975), Shafer (1979), Zadeh (1975), etc. Up to now, fuzzy logic is still a popular research area in theory and applications (see Elmas, Deperlioglu, \& Sayan, 2009; Ma, Zhang, Yan, \& Cheng, 2011). Multi-valued logic has been well developed, but the interpretation of the truth value is controversial. In 1976, Nilsson (1986) considered the truth value as a probability and initiated probability logic. Recently, Li and Liu (2009) introduced credibilistic logic to deal with fuzzy knowledge. Furthermore, Li and Liu (2009) proposed uncertain logic, which can be seen as a generalization of all these multi-valued logics, in which the truth value is defined as the uncertain measure that the proposition is true. One advantage of uncertain logic is its consistency with classical logic. In addition, uncertain inference was proposed by Liu (2009) as a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty.

The purpose of this paper is to carry out further research in uncertain logic. A formula for computing the truth value of uncertain propositions is proved. The rest of this paper is organized as follows: some basic concepts of uncertainty theory are recalled in Section 2. Several basic definitions of uncertain logic are contained in Section 3. A formula for computing the truth value of uncertain propositions is proved and some examples are calculated in Section 4. Finally, a brief summary is given in Section 5.

\section{2. Preliminaries}

Let $F$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra over $F$. Each element $A \in \mathcal{L}$ is called an event.

\begin{definition} (Liu (2007)). The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following four axioms:
\end{definition}
Axiom 1 (Normality). $M(\emptyset) = 1$.

Axiom 2 (Self-duality). $M(A) + M(A^c) = 1$ for any event $A$.

Axiom 3 (Countable subadditivity). For every countable sequence of events $\{A_i\}$, we have

$$M\left(\bigcap_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} M(A_i).$$

Axiom 4 (Product measure axiom). Let $\Gamma_k$ be nonempty sets on which $M_k$ are uncertain measures, $k = 1, 2, \ldots, n$, respectively. Then the product uncertain measure $M$ is an uncertain measure on the product $\sigma$-algebra $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying

$$M\left(\prod_{k=1}^{n} A_k \right) = \min_{1 \leq i \leq n} M_k(A_k).$$

An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers.

Definition 2 (Liu (2007)). The uncertain variables $X_1, X_2, \ldots, X_n$ are said to be independent if

$$M(\bigcap_{i=1}^{n} X_i \in B_i) = \min_{1 \leq i \leq n} M(X_i \in B_i)$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 1 (Liu (2009), Operational law). Let $\xi, \eta, \zeta$ be independent uncertain variables, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a measurable function. Then $\zeta = f(\xi, \eta, \zeta)$ is an uncertain variable such that

$$M(\zeta \in B) = \left\{ \begin{array}{ll}
\sup_{f(B_1, B_2, B_3) \in B} \min M_1(B_1), & \text{if } \sup_{f(B_1, B_2, B_3) \in B} \min M_2(B_2) > 0.5 \\
1 - \sup_{f(B_1, B_2, B_3) \in B} \min M_3(B_3), & \text{if } \sup_{f(B_1, B_2, B_3) \in B} \min M_3(B_3) > 0.5 \\
0.5, & \text{otherwise}
\end{array} \right.$$

3. Uncertain logic

Uncertain logic was designed by Li and Liu (2009) as a generalization of classical logic. Liu gave the definition of uncertain proposition which is a statement with truth value in $[0, 1]$. In fact, the uncertain proposition $X$ is essentially an uncertain variable taking values 0 or 1, where $X = 1$ means $X$ is true and $X = 0$ means $X$ is false. Uncertain propositions are called independent if they are independent uncertain variables.

An uncertain formula is a finite sequence of uncertain propositions and connective symbols that are well defined. If $\zeta, \eta, \tau$ are uncertain propositions, then $X = \neg \zeta, X = \zeta \land \eta, X = (\zeta \lor \eta) \rightarrow \tau$ are all uncertain formulas. Uncertain formulas are called independent if they are independent uncertain variables.

Truth value is a key concept in uncertain logic and is defined as the uncertain measure that the uncertain formula is true.

Definition 3 (Li and Liu (2009)). Let $X$ be an uncertain formula. Then the truth value of $X$ is defined as the uncertain measure that the uncertain formula $X$ is true, i.e.,

$$T(X) = M(X) = 1.$$

4. A new formula for truth value

Theorem 2 (Truth value theorem). Let $X$ be an uncertain formula containing independent uncertain propositions $\xi_1, \xi_2, \ldots, \xi_n$ whose truth function is $f$. Then the truth value of $X$ is

$$T(X) = \left\{ \begin{array}{ll}
\sup_{f(B_1, B_2, \ldots, B_n) \in B} \min M_1(B_1), & \text{if } \sup_{f(B_1, B_2, \ldots, B_n) \in B} \min M_2(B_2) > 0.5 \\
1 - \sup_{f(B_1, B_2, \ldots, B_n) \in B} \min M_3(B_3), & \text{if } \sup_{f(B_1, B_2, \ldots, B_n) \in B} \min M_3(B_3) > 0.5 \\
0.5, & \text{otherwise}
\end{array} \right.$$
Thus we have
\[
\sup_{f(b_1, b_2, \ldots, b_n) = 0} \min_{1 \leq i \leq n} M \{ \xi_i \in B_i \} \geq 0.5.
\]

Thus
\[
T(X) = 0.5 = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} p_i(X_i).
\]

If
\[
\sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} p_i(X_i) < 0.5,
\]

by Case 1 we have
\[
T(X) = 1 - \sup_{f(x_1, x_2, \ldots, x_n) = 0} \min_{1 \leq i \leq n} p_i(X_i).
\]

The theorem is proved. \(\square\)

**Example 1.** Suppose that \(\xi\) and \(\eta\) are independent uncertain propositions with truth values \(\alpha\) and \(\beta\), respectively. Then
\[
X = \text{“both } \xi \text{ and } \eta \text{ are true” or “both } \xi \text{ and } \eta \text{ are false”}
\]
is an uncertain formula. It is obvious that the truth function of \(X\) is
\[
f(1, 1) = 1, f(0, 0) = 1, f(1, 0) = 0, f(0, 1) = 0.
\]

Let us calculate its truth value \(T(X)\) by the truth value theorem. When \(\alpha \geq 0.5\) and \(\beta \geq 0.5\), we have
\[
\sup_{i=1}^{n} p_i(X_i) = \max\{a \land \beta, (1 - \alpha) \land (1 - \beta)\} = a \land \beta \geq 0.5.
\]

It follows the truth value theorem that
\[
T(X) = 1 - \sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq n} p_i(X_i) = 1 - \max\{(1 - \alpha) \land \beta, a \land (1 - \beta)\}
\]
\[
= a \land \beta.
\]

When \(\alpha \geq 0.5\) and \(\beta < 0.5\), we have
\[
\sup_{i=1}^{n} p_i(X_i) = \max\{a \land \beta, (1 - \alpha) \land (1 - \beta)\} = (1 - \alpha) \lor \beta \leq 0.5.
\]

It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = (1 - \alpha) \lor \beta.
\]

When \(\alpha < 0.5\) and \(\beta \geq 0.5\), we have
\[
\sup_{i=1}^{n} p_i(X_i) = \max\{a \land \beta, (1 - \alpha) \land (1 - \beta)\} = (1 - \alpha) \land (1 - \beta) \leq 0.5.
\]

It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = (1 - \alpha) \land (1 - \beta).
\]

When \(\alpha < 0.5\) and \(\beta < 0.5\), we have
\[
\sup_{i=1}^{n} p_i(X_i) = \max\{a \land \beta, (1 - \alpha) \land (1 - \beta)\} = a \lor (1 - \beta) \leq 0.5.
\]

Thus we have
\[
T(X) = 1 - \sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq n} p_i(X_i) = 1 - \max\{(1 - \alpha) \land \beta, a \land (1 - \beta)\}
\]
\[
= 1 - a \lor (1 - \alpha) \land (1 - \beta)
\]

Thus we have
\[
T(X) = \begin{cases} 
\alpha \land \beta, & \text{if } \alpha \geq 0.5 \text{ and } \beta \geq 0.5 \\
(1 - \alpha) \lor \beta, & \text{if } \alpha \geq 0.5 \text{ and } \beta < 0.5 \\
\alpha \lor (1 - \beta), & \text{if } \alpha < 0.5 \text{ and } \beta \geq 0.5 \\
(1 - \alpha) \land (1 - \beta), & \text{if } \alpha < 0.5 \text{ and } \beta < 0.5.
\end{cases}
\]

**Example 2.** Suppose that \(\xi\) and \(\eta\) are independent uncertain propositions with truth values \(\alpha\) and \(\beta\), respectively. Then
\[
X = \text{“one of } \xi \text{ and } \eta \text{ is true” and “one of } \xi \text{ and } \eta \text{ is false”}
\]
is an uncertain formula. It is obvious the truth function of \(X\) is
\[
f(1, 1) = 0, f(0, 0) = 0, f(1, 0) = 0, f(0, 1) = 1.
\]

Let us calculate its truth value \(T(X)\) by the truth value theorem. When \(\alpha \geq 0.5\) and \(\beta \geq 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = \max\{a \land (1 - \beta), (1 - \alpha) \land \beta\}
\]
\[
= (1 - \alpha) \lor (1 - \beta) \leq 0.5.
\]

It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = (1 - \alpha) \lor (1 - \beta).
\]

When \(\alpha \geq 0.5\) and \(\beta < 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = \max\{a \land (1 - \beta), (1 - \alpha) \land \beta\} = a \land (1 - \beta) \geq 0.5.
\]

It follows from the truth value theorem that
\[
T(X) = 1 - \sup_{f(x_1, x_2) = 0} \min_{1 \leq i \leq n} p_i(X_i) = 1 - (1 - \alpha) \lor (1 - \beta).
\]

When \(\alpha < 0.5\) and \(\beta \geq 0.5\), we have
\[
\sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = \max\{a \land (1 - \beta), (1 - \alpha) \land \beta\} = a \lor (1 - \beta) \leq 0.5.
\]

It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2) = 1} \min_{1 \leq i \leq n} p_i(X_i) = a \lor (1 - \beta).
\]

Thus we have
\[
T(X) = \begin{cases} 
(1 - \alpha) \lor (1 - \beta), & \text{if } \alpha \geq 0.5 \text{ and } \beta \geq 0.5 \\
a \land (1 - \beta), & \text{if } \alpha \geq 0.5 \text{ and } \beta < 0.5 \\
(1 - \alpha) \land \beta, & \text{if } \alpha < 0.5 \text{ and } \beta \geq 0.5 \\
a \lor (1 - \beta), & \text{if } \alpha < 0.5 \text{ and } \beta < 0.5.
\end{cases}
\]

**Example 3.** Suppose that \(\xi_1, \xi_2, \ldots, \xi_n\) are independent uncertain propositions with truth values \(x_1, x_2, \ldots, x_n\), respectively. For any integer \(k \geq 1\) with \(1 \leq k \leq n\),
\[
X_k = \text{“at least } k \text{ propositions of } \xi_1, \xi_2, \ldots, \xi_n \text{ are true”}
\]
is an uncertain formula. The truth function of \(X_k\) is
\[
f(x_1, x_2, \ldots, x_n) = \begin{cases} 
1, & \text{if } x_1 + x_2 + \ldots + x_n \geq k \\
0, & \text{if } x_1 + x_2 + \ldots + x_n < k.
\end{cases}
\]
The truth value theorem may produce the truth value \( T(X) \). Without loss of generality, we assume \( x_1 \geq x_2 \geq \cdots \geq x_n \). When \( x_k < 0.5 \), we have
\[
\sup_{f(x_1, x_2, \ldots, x_k)} \min_{1 \leq i \leq n} v_i(x_i) = x_k < 0.5.
\]
It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2, \ldots, x_k)} \min_{1 \leq i \leq n} v_i(x_i) = x_k.
\]
When \( x_k \geq 0.5 \), we have
\[
\sup_{f(x_1, x_2, \ldots, x_k)} \min_{1 \leq i \leq n} v_i(x_i) \geq x_k \land \min_{r \leq k \leq n} (x_r \lor (1 - x_i)) \geq 0.5.
\]
It follows from the truth value theorem that
\[
T(X) = 1 - \sup_{f(x_1, x_2, \ldots, x_k) \leq 0} \min_{1 \leq i \leq n} v_i(x_i) = 1 - (1 - x_k) = x_k.
\]
Thus we always have \( T(X) = x_k \). That is \( T(X) = \) the \( k \) th largest value of \( x_1, x_2, \ldots, x_n \).

When \( k = 1 \), it is clear that \( T(X) \) becomes the largest value, i.e.,
\[
(T(X))^\land = x_1 \lor x_2 \lor \cdots \lor x_n.
\]
When \( k = n \), it is clear that \( T(X) \) becomes the smallest value, i.e.,
\[
(T(X))^\lor = x_1 \land x_2 \land \cdots \land x_n.
\]

**Example 4.** Suppose that \( \xi, \eta, \) and \( \zeta \) are independent uncertain propositions with truth values \( x, \beta \) and \( \gamma \), respectively. Then
\[
T(X) = (\xi \lor \eta) \rightarrow \zeta
\]
is an uncertain proposition. It is obvious that the truth function of \( X \) is
\[
f(1, 1, 1) = 1, \quad f(1, 0, 1) = 1, \quad f(0, 1, 1) = 1, \quad f(0, 0, 1) = 1, \quad f(1, 1, 0) = 0, \quad f(1, 0, 0) = 0, \quad f(0, 1, 0) = 0, \quad f(0, 0, 0) = 1.
\]
Let us calculate its truth value \( T(X) \) by the truth value theorem. When \( x \geq 0.5, \beta \geq 0.5 \) and \( \gamma \geq 0.5 \), we have
\[
\sup_{f(x,y,z)\geq1} v_i(x_i) \geq \min(x, \beta, \gamma) \geq 0.5.
\]
It follows from the truth value theorem that
\[
T(X) = 1 - \sup_{f(x_1, x_2, \ldots, x_k) \geq 0} \min_{1 \leq i \leq n} v_i(x_i)
\]
\[
= 1 - \max((1 - \gamma) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta)) = \gamma.
\]
When \( \gamma \geq 0.5, \beta < 0.5 \) and \( \gamma < 0.5 \), we have
\[
\sup_{f(x_1, x_2, \ldots, x_k) \geq 1} v_i(x_i)
\]
\[
= \max(\beta \land \gamma, (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta))
\]
\[
= (1 - \beta) \land \gamma \leq 0.5.
\]
It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2, \ldots, x_k) \geq 1} \min_{1 \leq i \leq n} v_i(x_i)
\]
\[
= (1 - \beta) \land \gamma.
\]
When \( \gamma \geq 0.5, \beta < 0.5 \) and \( \beta < 0.5 \), we have
\[
\sup_{f(x_1, x_2, \ldots, x_k) \geq 1} v_i(x_i)
\]
\[
= \sup_{f(x_1, x_2, \ldots, x_k) \geq 1} \min_{1 \leq i \leq n} v_i(x_i)
\]
\[
= \min((1 - \gamma) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta) \land (1 - \beta)) = 0.5.
\]
It follows from the truth value theorem that
\[
T(X) = \sup_{f(x_1, x_2, \ldots, x_k) \geq 1} \min_{1 \leq i \leq n} v_i(x_i)
\]
\[
= (1 - \beta) \land \gamma.
\]
Thus we have
\[
T(X) = (1 - \beta) \land \gamma.
\]

**5. Conclusion.**

Uncertain logic was defined within the framework of uncertainty theory as a generalization of classical logic. One advantage of uncertain logic is its consistency with classical logic. As our main result, a formula for truth value of uncertain propositions was proposed. Finally, some examples were given.
Acknowledgments

This work was supported by National Natural Science Foundation of China Grant No. 60874067 and No.91024032. Dan A. Ralescu’s work was partly supported by a Taft Travel for Research Grant.

References