A Note on Uncertain Sequence Convergence

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Abstract

In this paper, an uncertain version of Borel-Cantelli lemma is proved. Besides, some convergence theorems of continuous uncertain measure are proposed. Using these theorems, it is proved that the uncertainty distribution function of the uncertain integral on a real function is a normal one.

Keywords: uncertainty theory; uncertain measure; uncertain variable; uncertain integral.

1 Introduction

Most human decisions are made in the state of uncertain environment. The performance of different uncertainty can be represented by a particular measure. Probability measure is a type of classic measure founded by Kolmogorov to study randomness one class of objective uncertainty. Besides randomness, fuzziness is a basic type of subjective uncertainty was initiated by Zadeh [15] via membership function in 1965. From then on many researchers studied fuzziness using measure such as fuzzy measure [13] possibility measure [16], credibility measure [5]. However, a lot of surveys showed that imprecise quantities represented in human language behave neither like randomness nor like fuzziness. In order to develop a more general measure to model imprecise quantities, Liu [6] founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms.


Since sequence convergence plays an important role in the fundamental theory of mathematics, many convergence properties of uncertain measure have been researched by Liu [6]. Besides, You [14] introduced the concept of convergence uniformly almost surely and showed the relationship among convergence almost surely, convergence in measure and convergence uniformly almost surely. Besides, Gao [2] introduced the concept of continuous uncertain measure and gave some properties of uncertain measure. In
this paper, we will first prove the Borel-Cantelli lemma in uncertain measure. With the assumption that
the uncertain sequence defined on continuous uncertain space, we will study the relationship between con-
vergence almost surely and convergence uniformly almost surely. Using these theorems, we will proved
that the distribution of uncertain integral on real function is still a normal uncertainty distribution.

The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are
recalled in Section 2. Several convergence theorems are proposed in section 3. The distribution function
of uncertain integration on a real function is studied in section 4. Finally, a brief summary is given in
Section 5.

2 Preliminary

Let $\Gamma$ be a nonempty set, and $L$ a $\sigma$-algebra over $\Gamma$. The uncertain measure $M$ (Liu[6]) is a set function
defined on $L$ satisfying the following five axioms:

Axiom 1. (Normality) $M\{\Gamma\} = 1$;

Axiom 2. (Monotonicity) $M\{\Lambda_1\} \leq M\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$;

Axiom 3. (Self-Duality) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$;

Axiom 4. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$ 

Axiom 5. (Product Measure Axiom) Let $\Gamma_k$ be nonempty sets on which $M_k$ are uncertain measures,
$k = 1, 2, \cdots, n$, respectively. Then the product uncertain measure $M$ is an uncertain measure on the
product $\sigma$-algebra $L = L_1 \times L_2 \times \cdots \times L_n$, satisfying

$$M\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1 \leq i \leq n} M_k\{\Lambda_k\}.$$ 

An uncertain variable is a measurable function from an uncertainty space $(\Gamma, L, M)$ to the set of real
numbers. The distribution of the uncertain variable $\xi$ is $\Phi(x) = M\{\gamma \in \Gamma|\xi(\gamma) \leq x\}$. The expected value
of an uncertain variable $\xi$ was defined by Liu [6] as

$$E[\xi] = \int_{0}^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx.$$ 

In order to study the properties of uncertain sequences, Liu [6] introduced the following four conver-
genence concepts.

An uncertain sequence $\{\xi_n\}$ is said to be convergent almost surely (a.s.) to an uncertain variable $\xi$ if
there exists an event $\Lambda$ with $M\{\Lambda\} = 1$ such that

$$\lim_{n \to \infty} |\xi_n(\gamma) - \xi(\gamma)| = 0$$

for every $\gamma \in \Lambda$.

An uncertain sequence $\{\xi_n\}$ is said to be convergent in measure to an uncertain variable $\xi$ if

$$\lim_{n \to \infty} M\{\gamma \in \Gamma| |\xi_n(\gamma) - \xi(\gamma)| \geq \varepsilon\} = 0$$

for each $\varepsilon > 0$. 

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Let $\xi, \xi_1, \xi_2, \cdots$ be uncertain variables with finite expected values. Then the sequence $\{\xi_n\}$ is called convergence in mean to $\xi$ if
\[ \lim_{n \to \infty} E[|\xi_n(\gamma) - \xi(\gamma)|] = 0. \]

Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. The sequence $\{\xi_n\}$ is called convergence in distribution to $\xi$ if $\Phi_n \to \Phi$ at any continuous point of $\Phi$.

Another type of convergence named convergence uniformly almost surely was introduced by You [14]. The sequence $\{\xi_n\}$ is said to be convergence uniformly a.s. to $\{\xi\}$ if there exists $E_k, M\{E_k\} \to 0$ such that $\xi_n$ converges uniformly to $\xi$ in $\Gamma - E_k$, for any fixed $k$. The relationship of the five convergence is as follows.

<table>
<thead>
<tr>
<th>Convergence Uniformly</th>
<th>$\Rightarrow$</th>
<th>Convergence</th>
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<tbody>
<tr>
<td>Almost Surely</td>
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<td>Almost Surely</td>
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<tr>
<td>Convergence in Mean</td>
<td>$\Rightarrow$</td>
<td>Convergence in Measure $\Rightarrow$ Convergence in Distribution</td>
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Table 1: Relationship among Convergence Concepts

3 Convergence Theorems

Suppose that $\Lambda_n$ is a sequence of subsets of $\Gamma$, we let
\[ \limsup_{n \to \infty} \Lambda_n = \lim_{n \to \infty} \bigcup_{m=n}^{\infty} \Lambda_m = \{\gamma \in \Gamma| \text{ infinite many } \Lambda_n \text{ contain } \gamma\} \]
and let
\[ \liminf_{n \to \infty} \Lambda_n = \lim_{n \to \infty} \bigcap_{m=n}^{\infty} \Lambda_m = \{\gamma \in \Gamma| \text{ only finite many } \Lambda_n \text{ do not contain } \gamma\}. \]

It has been proved by You [14] that the sequence $\xi_n$ converges a.s. to $\xi$ if and only if for any $\epsilon > 0$, we have
\[ M\left\{ \limsup_{n \to \infty} \bigcup_{m=n}^{\infty} \{\gamma \in \Gamma| ||\xi_m(\gamma) - \xi(\gamma)|| \geq \epsilon\} \right\} = 0. \]

You [14] also proved that the sequence $\xi_n$ converges uniformly a.s. to $\xi$ if and only if
\[ \lim_{n \to \infty} M\left\{ \bigcup_{m=n}^{\infty} \{\gamma \in \Gamma| ||\xi_m(\gamma) - \xi(\gamma)|| \geq \epsilon\} \right\} = 0. \]

Now we will prove the Borel-Cantelli Lemma in uncertainty measure.

**Theorem 1.** Suppose that $\Lambda_n$ is a sequence of subsets of $\Gamma$. If $\sum_{i=1}^{\infty} M\{A_i\} < \infty$, then $M\{\limsup \Lambda_n\} = 0$.

**Proof:** The sequence of events $\bigcup_{i=n}^{\infty} A_i$ is decreasing in $n$ and converges to $\cap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Then
\[ 0 \leq M\left\{ \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right\} = M\left\{ \lim_{n \to \infty} \bigcup_{i=n}^{\infty} A_i \right\} \leq \lim_{n \to \infty} \sum_{i=n}^{\infty} M\{A_i\} = 0. \]

The theorem is proved.

Next we will deal with continuous uncertain measure. The definition of continuous uncertain measure is introduced the following.
Definition 1. (Gao [2]) An uncertain measure $\mathcal{M}$ is said continuous if for any sequence of events $\{\Lambda_i\}$ with $\lim i \to \infty$ exists, we have $\mathcal{M}\{ \lim_{i \to \infty} \Lambda_i \} = \lim_{i \to \infty} \mathcal{M}\{\Lambda_i\}$.

Gao [2] proved that the uncertain measure $\mathcal{M}$ is continuous is equivalent to $\mathcal{M}$ is continuous from above, $\mathcal{M}$ continuous from above at $\emptyset$, $\mathcal{M}$ is continuous from below, $\mathcal{M}$ is continuous from below at $\Gamma$. Next, we will study the convergence properties of sequences defined on continuous uncertain space.

Theorem 2. Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on the same continuous uncertainty space. If $\{\xi_n\}$ converges a.s. to $\xi$, then $\{\xi_n\}$ converges uniformly a.s. to $\xi$.

Proof: The sequence $\xi_n$ converges a.s. $\xi$ if and only if for any $\epsilon > 0$, we have

$$\mathcal{M}\left\{ \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma | |\xi_n(\gamma) - \xi(\gamma)| \geq \epsilon \} \right\} = 0.$$ 

The event

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma | |\xi_n(\gamma) - \xi(\gamma)| \geq \epsilon \} = \lim_{m \to \infty} \left( \bigcup_{n=m}^{\infty} \{ \gamma \in \Gamma | |\xi_n(\gamma) - \xi(\gamma)| \geq \epsilon \} \right).$$

Since the uncertain variables $\xi_n$ are defined on the continuous uncertainty space, we have

$$\lim_{m \to \infty} \mathcal{M}\left\{ \bigcup_{n=m}^{\infty} \{ \gamma \in \Gamma | |\xi_n(\gamma) - \xi(\gamma)| \} \right\} = \mathcal{M}\left\{ \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma | |\xi_n(\gamma) - \xi(\gamma)| \} \right\} = 0.$$ 

Thus $\{\xi_n\}$ converges uniformly a.s. to $\xi$.

Theorem 3. Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on continuous uncertainty space. If $\{\xi_n\}$ converges a.s. to $\xi$, then $\{\xi_n\}$ converges to $\xi$ in uncertain measure.

Proof: It has been proved that convergence uniformly a.s. implies convergence in uncertain measure. The theorem is easily proved through the above theorem.

Theorem 4. Suppose $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on continuous uncertainty space. If $\{\xi_n\}$ converges a.s. to $\xi$, then $\{\xi_n\}$ converges to $\xi$ in distribution.

Proof: It is easily proved by the fact that convergence uniformly a.s. implies convergence in distribution when uncertain measure is continuous.

Theorem 5. Suppose that $\xi, \xi_1, \xi_2, \cdots$ are uncertain variables defined on continuous uncertainty space. Then $\xi_n$ converges in measure to $\xi$ if and only if there exists subsequence $\{\xi_{n'_k}\}$ of $\{\xi_n\}$ such that $\xi_{n'_k}$ converges a.s. to $\xi$, ($k \to \infty$), for any subsequence $\{\xi_{n'}\}$ of $\{\xi_n\}$

Proof: It is easily proved by the fact that convergence uniformly a.s. implies convergence a.s. when the uncertain measure is continuous.

4 Uncertain Integral

An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution function

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right), \quad -\infty < x < +\infty$$

denoted by $\mathcal{N}(e, \sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$. It is easily to calculate that the expected value of $\xi$ $E[\xi]$ = $e$ and variance $V[\xi]$ = $\sigma^2$. Let $\xi$ and $\eta$ be independent normal uncertain variables with expected values $e_1$ and $e_2$, variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Then the uncertain variable $a_1\xi + a_2\eta$ is also normal with expected value $a_1e_1 + a_2e_2$ and variance $(|a_1|\sigma_1^2 + |a_2|\sigma_2^2)$ for any real numbers $a_1$ and $a_2$. 

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Definition 2. (Liu [8]) An uncertain process $C_t$ is said to be a canonical process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$,
whose uncertainty distribution is
\[ \Phi(x) = \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3} t} \right) \right)^{-1}, \quad x \in \mathbb{R}. \]

Definition 3. (Liu [8]) Let $X_t$ be an uncertain process and let $C_t$ be a canonical process. For any
partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{m+1} = b$, the mesh is written as
\[ \Delta = \max_{1 \leq i \leq m} |t_{i+1} - t_i|. \]
Then the uncertain integral of $X_t$ with respect to $C_t$ is
\[ \int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \]
provided that the limit exists almost surely and is an uncertain variable.

Theorem 6. Let $C_t$ be a canonical process defined on continuous uncertain space and let $f(t)$ be a
deterministic and integrable function with respect to $t$. Then the uncertain integral
\[ \int_0^s f(t) dC_t \]
is a normal uncertain variable at each time $s$, i.e.,
\[ \int_0^s f(t) dC_t \sim N \left( 0, \int_0^s |f(t)| dt \right). \]

Proof: For any partition of closed interval $[0, s]$ with $0 = t_1 < t_2 < \cdots < t_{k+1} = s$ and the mesh size $\Delta$, the uncertain integral is
\[ \lim_{\Delta \to 0} \sum_{i=0}^{m} f(t_i)(C_{t_{i+1}} - C_i) = \int_0^s f(t) dC_t \quad (a.s.) \]
Since the increments of canonical process $C_{i+1} - C_i$ $(i = 1, 2, \cdots, m)$ are independent normal uncertain variables., the sum $\sum_{i=0}^{m} f(t_i)(C_{i+1} - C_i)$ is also a normal uncertain variable with the expected value 0 and
variance $(\sum_{i=0}^{m} |f(t_i)| (t_{i+1} - t_i))^2$. Under the assumption that the canonical process is defined on continuous
uncertain space and by theorem 3, we have
\[ \mathbb{M} \left\{ \int_0^s f(t) dC_t \leq x \right\} = \lim_{m \to \infty} \mathbb{M} \left\{ \sum_{i=0}^{m} f(t_i)(C_{i+1} - C_i) \leq x \right\} = \lim_{m \to \infty} \mathbb{M} \left\{ \sum_{i=0}^{m} f(t_i)(C_{i+1} - C_i) \right\} = \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3} \sum_{i=0}^{m} |f(t_i)| (t_{i+1} - t_i)} \right) \right)^{-1}
= \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3} \int_0^s |f(t)| dt} \right) \right)^{-1}
\]
The theorem is proved.
5 Conclusion

In this paper, an uncertain version of Borel-Cantelli lemma was proved. Under the assumption that the uncertain measure is continuous, several convergence theorems were proposed. Based on these theorems, the property that the uncertainty distribution of the uncertain integral on a real function is a normal one was studied.

Acknowledgments

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References


