Existence and Uniqueness of Solution for Uncertain Differential Equations with Non-Lipschitz Coefficients

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Abstract: This paper studies a class of uncertain differential equations with non-Lipschitz coefficients. The existence and uniqueness of solutions for uncertain differential equations with non-Lipschitz coefficients is first proved. Then the dependence of uncertain differential equations on initial values is discussed. Finally, a pathwise uniqueness of solutions is obtained. Our main tool is the Gronwall lemma.

Key words: Uncertain differential equation; Canonical Process; Existence and uniqueness; Non-Lipschitz coefficients; Gronwall lemma

1. Introduction

The real life decisions are usually made in the state of uncertainty. To describe a set without definite boundary, fuzzy set was initiated by [19] in 1965, whose membership function indicates the degree of an element belonging to it. In order to measure a fuzzy event, a self-duality credibility measure was introduced by Liu and Liu [16] in 2002. Since an axiomatic foundation for credibility theory was constructed by Liu [13] and a sufficient and necessary condition is given by Li and Liu [17].

Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system. In order to describe this phenomena, a fuzzy random variable was introduced by Puri and Ralescu [18] as a random element taking fuzzy variable values. The theory on fuzzy random variables has be investigated by many authors e.g. Fei [2]−[7], Fei and Wu [8], Fei et al. [9, 10], Feng [11] and Li and Ogura [15] etc.. In 2004, Liu [13] introduced the concept of fuzzy random variables which is different from the above one.

It is well known that the additivity axiom of classical measure theory has been challenged by many mathematicians. The earliest challenge was from the theory of capacities by Choquet [1] in which monotonicity and continuity axioms were assumed. For this reason, Liu [13, 14] founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, and countable sub-additivity axioms. In this paper, we will introduce uncertain differential equations in the framework of Liu’s uncertainty space which are the counterparts of classical stochastic differential equations (cf. Ikeda and Watanabe [12]).

Our purpose in this article is to discuss the behavior of solutions to the Cauchy problem, such as uniqueness of solutions and noncontactness etc. which are important in the theory of uncertain dynamical system analysis. Therefore, the results obtained by Fei[7] in the theory of credibility are ex-
tended.

The organization of the paper is as follows. Section 2 is preliminaries and Lemmas. In section 3 the existence and uniqueness of solution for uncertain differential equation are studied. Finally, the conclusions are placed in section 4.

2. Preliminaries

Throughout this paper, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space described in Liu [14], the sub-$\sigma$-field family $\{\mathcal{L}_t, t \in \mathbb{R}_+\}$ of $\mathcal{L}$ satisfies the usual condition, and $\mathcal{L}_s \subseteq \mathcal{L}_t \subseteq \mathcal{L}$ for $0 \leq s < t < \infty$. We set $\mathcal{L}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{L}_t)$. Thus we equip the uncertainty space $(\Gamma, \mathcal{M})$ with a filtration. As classical stochastic calculus, in the investigation of uncertain processes, there is a very important, nontechnical reason to include $\sigma$-fields, and that is to keep track of information. The temporal feature of a stochastic process suggests a flow of time, in which, at every moment $t \geq 0$, we can talk about a pass, present, and future and can ask how much an observer of the process knows about it at present, as compared to how much he knew at some point in past or will know at some point in future.

Given a uncertain process $X = \{X(t); 0 \leq t \leq \infty\}$, the simplest choice of a filtration is that generated by the process itself, i.e.,

$$\mathcal{L}_t^X \triangleq \sigma(X(s); 0 \leq s \leq t),$$

the smallest $\sigma$-field with respect to which $X(s)$ is measurable for every $s \in [0, t]$. We interpret $\Lambda \in \mathcal{L}_t^X$ to mean that by time $t$, an observer of $X$ knows whether or not $\Lambda$ has occurred.

The uncertain process $X$ is adapted to the filtration $\{\mathcal{L}_t\}$ if, for each $t \geq 0$, $X(t)$ is an $\mathcal{L}_t$-measurable random variable.

A uncertain time $T$ is an uncertainty stopping time of the filtration, if the event $\{T \leq t\}$ belongs to the $\sigma$-field $\mathcal{L}_t$, for every $t \geq 0$. A filtration $\{\mathcal{L}_t\}$ is said to satisfy the usual conditions if it is right-continuous and $\mathcal{L}_0$ contains all the $\mathcal{M}$-negligible events in $\mathcal{L}$.

Before we define the uncertain differential equations, we define uncertain integral. Here, all uncertain processes involved are $\mathcal{L}_t$-adapted ones.

**Definition 2.1.** Let $X(t)$ be an uncertain process and let $C(t)$ be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} (t_{i+1} - t_i).$$

Then the uncertain integral of $X(t)$ with respect to $C(t)$ is

$$\int_a^b X(t)dC(t) = \lim_{\Delta \to 0} \sum_{i=1}^k X(t_i)(C(t_{i+1}) - C(t_i))$$

provided that the limit exists almost surely and is an uncertain variable.

In what follows, in the remainder of the paper we discuss the one-dimensional uncertain differential equation (cf. Liu[14])

$$dX(t) = f(X(t))dt + g(X(t))dC(t),$$

$$X(0) = x_0, \ t \geq 0,$$  \hspace{1cm} (2.1)

where $C(t)$ is one-dimensional canonical process defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M}; \mathcal{L}_t)$ with a filtration $\{\mathcal{L}_t\}_{t \geq 0}$ satisfying the usual condition, $x_0$ is a uncertain variable. Note that canonical process can be referred to Definition 4.1 of Liu [14] and the results of one-dimensional case can be generalized to the $n$-dimensional case.

For the our aim, we provide the following lemmas similar to lemma 2.2 in Fei[7] for the case of credibility. The following Lemma 2.2 is easily obtained.

**Lemma 2.2.** (i) Let $C(t)$ be an uncertain process. For any given $\gamma$ with $\mathcal{M}\{\gamma\} > 0$, the path $C(t, \gamma)$ is
Lipschitz continuous, that is, the following inequality holds

$$|C(t_1, \gamma) - C(t_2, \gamma)| < \eta(\gamma)|t_1 - t_2|,$$

where $\eta$ is an uncertain variable called the Lipschitz constant of an uncertain process with

$$\eta(\gamma) = \begin{cases} \sup_{0 \leq s < t} \frac{|C(t, \gamma) - C(s, \gamma)|}{t-s}, & \text{if } \mathcal{M}\{\gamma\} > 0, \\ \infty, & \text{otherwise}, \end{cases}$$

(2.2)

and $E[\eta^p] < \infty$, $\forall \ p > 0$.

(ii) Let $C(t)$ be a canonical process process, and let $h(t;c)$ be a continuously differentiable function. Define $X(t) = h(t;C(t))$. Then we have the following chain rule

$$dX(t) = \frac{\partial h(t;C(t))}{\partial t}dt + \frac{\partial h(t;C(t))}{\partial c}dC(t).$$

Lemma 2.3. Let $g(t)$ be continuous uncertain process, the following inequality of uncertain integral holds

$$\left| \int_a^b g(t)dC(t) \right| \leq \eta \int_a^b |g(t)|dt,$$

where $\eta = \eta(\gamma)$ is defined in (2.2).

Proof. Let $a = t_1 < \cdots < t_{k+1} = b, \Delta = \max_{1 \leq i \leq k}(t_{i+1} - t_i)$. By the definition of uncertain integral and Lemma 2.2 (i), we derive

$$\left| \int_a^b g(t)dC(t) \right| = \left| \lim_{\Delta \to 0} \sum_{i=1}^k g(t_i)(C(t_{i+1}) - C(t_i)) \right|$$

$$\leq \lim_{\Delta \to 0} \sum_{i=1}^k |g(t_i)| |C(t_{i+1}) - C(t_i)|$$

$$\leq \eta \lim_{\Delta \to 0} \sum_{i=1}^k |g(t_i)||t_{i+1} - t_i|$$

$$\leq \eta \int_a^b |g(t)|dt,$$

which shows the assertion is true. Thus the proof is complete. $\square$

3. Existence and uniqueness of solutions

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $X(t)$ be a solution of the Eq.(2.1) with the lifetime $\zeta(\gamma)$ defined by

$$\zeta(\gamma) \triangleq \inf\{t : |X(t;\gamma)| = +\infty\}.$$

The following theorem investigates uniqueness on solutions of uncertain differential equations.

**Theorem 3.1.** Let $r : \mathbb{R}_+ \to [0, +\infty)$ be a continuous function such that $\int_0^\infty ds/r(s+1) = +\infty$.

Assume that for some constant $H > 0$,

$$|f(x)| + |g(x)| \leq H(|x|^r|x|^2 + 1).$$

(3.1)

Then the lifetime of the solution $X(t, x_0)$ for Eq.(2.1) is infinite: $\zeta = +\infty$. That is

$$\lim_{|x_0| \to +\infty} |X(t, x_0)| = +\infty \quad a.s., \mathcal{M}.$$

Proof. We only need to prove the lifetime $\zeta = +\infty$ of solution for the uncertain differential equation (2.1). Define now for $\xi \geq 0$,

$$\psi(\xi) = \int_0^\xi \frac{ds}{s^r(s+1)}, \Phi(\xi) = e^{\psi(\xi)}.$$

We have

$$\Phi'(\xi) = \frac{\Phi(\xi)}{\xi^{r+1}}.$$

Let $\xi(t) = X^2(t)$, where $X(t)$ is a solution to Eq.(2.1). From the chain rule of Lemma 2.3, we have

$$d\xi(t) = 2X(t)f(t)dt + 2X(t)g(t)dC(t),$$

which implies

$$\Phi(\xi(t;\gamma)) = \Phi(\xi_0(\gamma)) + 2\int_0^t \Phi'(\xi(s))X(t;\gamma)f(X(t;\gamma))dt + 2\int_0^t \Phi'(\xi(s))X(t;\gamma)g(X(t;\gamma))dC(t;\gamma).$$
Hence, by Lemma 2.3 and the assumption (3.1) we obtain, for $\forall \gamma \in \Gamma$ with $\mathcal{M}\{\gamma\} > 0$,

$$
\Phi(\xi(t;\gamma)) \leq \Phi(\xi_0(\gamma)) + 2 \int_0^t \Phi'(\xi(s;\gamma))|X(t;\gamma) - f(X(t;\gamma))|dt + 2M\{\gamma\} \sup_{s \leq t} \Phi(\xi(s;\gamma)) \leq \Phi(\xi_0(\gamma)) + 2H(1 + \eta(\gamma)) \int_0^t \Phi(\xi(s;\gamma)) d\gamma \]

Thus the proof is complete. \(\square\)

Note that function

$$
r(s) = \left\{ \begin{array}{ll}
\log s, & s \geq e, \\
1, & 0 \leq s < e,
\end{array} \right.
$$

is a typical example satisfying the above condition.

In what follows, we shall study the uniqueness of Eq.(2.1).

For simplicity, we shall assume that the solutions to Eq.(2.1) are existent. We call Eq.(2.1) has a unique solution if $|X(t) - Y(t)| = 0$, a.s., for $\forall t \in [0, \infty)$, where $X(t)$ and $Y(t)$ are any two solutions to Eq. (2.1).

**Theorem 3.2** Let $r : (0, 1) \rightarrow [1, +\infty)$ be a continuous function such that

$$
\int_0^a \frac{ds}{sr(s)} = +\infty, \text{ for some } a \in (0, 1).
$$

Assume that for some constant $H > 0$,

$$
|f(x) - f(y)| + |g(x) - g(y)| \leq H|x - y| r(|x - y|),
$$

for $|x - y| < 1$. \(3.5\)

Then the uncertain differential equation (2.1) has a unique solution.

**Proof.** Let $\rho > 0$. Consider

$$
\psi_\rho(\xi) = \int_0^\xi \frac{ds}{sr(s) + \rho}, \quad \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.
$$

Hence, we have

$$
\Phi'(\xi) = \frac{\Phi_\rho(\xi)}{\xi r(\xi) + \rho}.
$$

Let $(X(t))_{t \geq 0}$ and $(Y(t))_{t \geq 0}$ be two solutions of Eq.(2.1). Set $\xi(t) = |X(t) - Y(t)|^2$. From the chain rule of Lemma 2.4, we have

$$
d\xi(t) = 2(X(t) - Y(t))f(X(t) - Y(t))dt + 2(X(t) - Y(t))g(X(t) - Y(t))dC(t),
$$
which implies
\[
\Phi_\rho(\xi(t)) = \Phi_\rho(\xi_0) + 2 \int_0^t \Phi_\rho'(\xi(s)) f(X(t) - Y(t)) \, dt + \int_0^t \Phi_\rho(\xi(s)) \, dt.
\]
Therefore, by Lemma 2.4 and the assumption (3.5) we obtain
\[
\Phi_\rho(\xi(t)) \leq \Phi_\rho(\xi_0) + 2 \int_0^t \Phi_\rho'(\xi(s)) |X(t) - Y(t)| \, dt + \int_0^t \Phi_\rho(\xi(s)) \, dt.
\]
Note that function \(\Phi_\rho(x) = e^{\rho x} - 1\) is convex and \(\Phi_\rho(0) = 0\).

Applying (3.8) for \((x_0, y_0)\), we get
\[
\Phi_\rho(\xi(t)) \leq e^{\rho_\tau x_0} e^{\rho \tau y_0}, \quad t < \tau(x_0, y_0).
\]
Thus the proof is complete. \(\square\)

Note that function
\[
\tau(s) = \begin{cases}
\log \frac{s}{1}, & 0 < s \leq \frac{1}{\epsilon}, \\
1, & s > \frac{1}{\epsilon},
\end{cases}
\]
is a typical example satisfying the above condition.

In what follows, we shall study the solution to uncertain differential equations on the dependence of initial values.

**Theorem 3.3** Assume that the conditions in Theorem 3.2 are satisfied. Then the image \(x_0 \to X_t(x_0)\) is continuous almost surely, uniformly with respect to \(t\) in any compact subset, where \(X_t(x_0) = X(t, x_0)\) is the solution to uncertain differential equation (2.1).

**Proof.** Let \(\varepsilon \in (0, 1)\). Consider a small parameter \(0 < \delta < \varepsilon\). Assume \(|x_0 - y_0| < \delta\). Consider
\[
\psi_\rho(x) = \int_0^\varepsilon \frac{ds}{s_r(s) + \rho}, \quad \Phi_\rho(x) = e^{\psi_\rho(x)}.
\]
Set \(\xi(t) = |X_t(x_0) - Y_t(y_0)|^2\). Define
\[
\tau(x_0, y_0) = \inf\{t > 0, \xi(t) \geq \varepsilon^2\}.
\]
As in proof of Theorem 3.2, we obtain for \(t < \tau(x_0, y_0)\), and some \(a.s.\mathcal{M}\) finite uncertain variable \(H > 0\),
\[
\Phi_\rho(\xi(t)) \leq \Phi_\rho(\xi(0)) e^{Ht}.
\]
Taking \(\rho = |x_0 - y_0|\), and noticing
\[
|\psi_\rho(\xi(0))| \leq \int_0^{\xi(0)} \frac{ds}{\rho} = \frac{\xi(0)}{\rho} = \rho
\]
which shows
\[
\Phi_\rho(\xi(0)) = e^{\psi_\rho(\xi(0))} \leq e^\rho,
\]
thus we obtain
\[
\Phi_\rho(\xi(t)) \leq e^\rho e^{Ht}, \quad t < \tau(x_0, y_0).
\]
Fix the point \(x_0\). If \(\lim_{y_0 \to x_0} \tau(x_0, y_0) = \tau < +\infty\), we can choose \(\lim_{n \to +\infty} y_n = x_0\) such that \(\lim_{n \to +\infty} \tau(x_0, y_n) = \tau\). Applying (3.8) for \((x_0, y_n)\) and letting \(t \uparrow \tau(x_0, y_n)\), we get
\[
\Phi_{\rho_n}(\varepsilon^2) = \Phi_{\rho_n}(\xi(\tau(x_0, y_n))) \leq e^{\rho_n e^{H\tau(x_0, y_n)}},
\]
where \(\rho_n = |x_0 - y_n|\). Letting \(n \to +\infty\), we have
\[
\lim_{n \to +\infty} \tau(x_0, y_n) = +\infty
\]
which causes a contraction. Hence we get \(\lim_{y_0 \to x_0} \tau(x_0, y_0) = +\infty\), which means that for any
Thus we complete the proof of the assertion. □

We call non confluence of two solutions $X_t(x_0)$ and $Y_t(y_0)$ for uncertain differential equation (2.1) with initial value $x_0$ and $y_0$, respectively if for any $t$, $|X_t(x_0) - X_t(y_0)| \neq 0$ a.s.$\mathcal{M}$. The following theorem gives the sufficient condition of the non confluence.

**Theorem 3.4** Assume that the conditions in Theorem 3.2 are satisfied. Then for $\forall x_0, y_0$ with $x_0 \neq y_0$, we have $X_t(x_0) \neq X_t(y_0)$ a.s.$\mathcal{M}$ for all $t \geq 0$.

**Proof.** Let $\xi(t) = |X_t(x_0) - X_t(y_0)|^2$. Without loss of generality, assume that $0 < \xi(0) < 1/2$. Let

$$
\tau = \inf \left\{ t > 0, \xi(t) \geq \frac{3}{4} \right\}.
$$

By starting from $\tau$ again, it is enough to prove that $\xi(t) > 0$ for $t < \tau$. Consider

$$
\psi_\rho(\xi) = \int_0^\xi \frac{ds}{s + \rho}, \Phi_\rho(\xi) = e^{\psi_\rho(\xi)}.
$$

By assumption (3.5), for $t < \tau$, as the proof in Theorem 3.2 we get that for some a.s.$\mathcal{M}$ finite constant $H > 0$,

$$
\Phi_\rho(\xi(t)) \geq \Phi_\rho(\xi(0)) - H \int_0^t \Phi_\rho(\xi(s))ds.
$$

Hence we have

$$
\Phi_\rho(\xi(t)) \geq \Phi_\rho(\xi(0))e^{-Ht}, \quad t < \tau.
$$

For $\rho > 0$ small enough, $\Phi_\rho(\xi(0))e^{-Ht} > 1$. It follows that $\Phi_\rho(\xi(t)) > 1$ or $\xi(t) > 0$. Thus the proof is complete. □

4. Conclusions

A pathwise uniqueness of solutions is obtained and the existence and uniqueness of solutions for uncertain differential equations with non-Lipschitz coefficients is proved, the dependence of uncertain differential equations on initial values is discussed.

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**References**


