American Option Pricing Formula for Uncertain Financial Market

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Abstract: Option pricing is the the core content of modern finance. American option is widely accepted by investors for its flexibility of exercising time. In this paper, American option pricing formula is calculated for uncertain financial market and some mathematical properties of them are discussed. In addition, some examples are proposed.

keywords: finance, uncertain process, option pricing

1 Introduction

Most human decisions are made in the state of uncertain environment. The performance of different uncertainty can be represented by a particular measure. Probability measure is a type of classic measure founded by Kolmogorov to study randomness one class of objective uncertainty. Besides randomness, fuzziness is a basic type of subjective uncertainty was initiated by Zadeh [20] via membership function in 1965. From then on many researchers studied such as possibility measure [21], credibility measure [7]. However, a lot of surveys showed that imprecise quantities represented in human language behave neither like randomness nor like fuzziness. In order to develop a more general measure to model imprecise quantities, Liu [8] founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, and countable subadditivity axioms. In order to provide a methodology for collecting and interpreting expert’s experimental data by uncertainty theory, uncertain statistics was started by Liu [12] in 2010 in which a questionnaire survey for collecting expert’s experimental data was designed and a principle of least squares for estimating uncertainty distributions was suggested. In addition, Wang and Peng [18] proposed a method of moments.


In the early 1970s, Black and Scholes [2] and, independently, Merton [15] constructed a theory for determining the stock option price which is the famous Black-Scholes formula. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. As a different doctrine, based on the assumption that stock price follows a geometric canonical process, uncertainty theory was first introduced into finance by Liu [10] in 2009. Furthermore, Liu [9] derived an uncertain stock model and a European option price formula. In addition, Peng [16] proposed a new uncertain stock model and other option price formulas. Besides, uncertainty theory was extended to insurance models by Liu [14] based on the assumption that the claim process is an uncertain renewal process.

Option pricing is the the core content of modern finance. American option is widely accepted by investors for its flexibility of exercising time. In this paper, American option pricing formula is calculated for uncertain financial market and some mathematical properties of them are discussed. The rest of the paper is organized as follows. Some preliminary concepts of uncertainty processes are recalled in Section 2. American option pricing formulae are derived and some properties of them are studied in section 3 and 4, respectively. Finally, a brief summary is given in Section 5.

2 Preliminary

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu [9] in 2008.

Definition 1. (Liu [9]) Let T be an index set and let \( \Gamma, \mathcal{L}, \mathcal{M} \) be an uncertainty space. An uncertain process is a measurable function from \( T \times (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for each \( t \in T \) and any Borel set \( B \) of real numbers, the...
Definition 2. An uncertain process $X_t$ is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where $t_0$ is the initial time and $t_1, t_2, \ldots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$.

Theorem 1. (Extreme Value Theorem, [12]) Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. Then the supremum

$$\sup_{0 \leq t \leq T} X_t$$

has an uncertainty distribution

$$\Phi(x) = \inf_{0 \leq t \leq T} \Phi_t(x).$$

Theorem 2. (Liu [12]) Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Phi_t(x)$ at each time $t$. If $f$ is a strictly increasing function, then the supremum

$$\sup_{0 \leq t \leq T} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0 \leq t \leq T} \Phi(f^{-1}(x)).$$

Theorem 3. (Liu [12]) Let $X_t$ be an independent increment process and have a continuous uncertainty distribution $\Psi_t(x)$ at each time $t$. If $f$ is a strictly decreasing function, then the supremum

$$\sup_{0 \leq t \leq T} f(X_t)$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0 \leq t \leq s} \Phi_t(f^{-1}(x)).$$

Definition 3. (Liu [10]) An uncertain process $C_t$ is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) $C_t$ has stationary and independent increments,

(iii) every increment $C_{t+s} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi x}{\sqrt{3t}}\right)\right)^{-1}, \; x \in \mathbb{R}.$$

If $C_t$ is canonical process, then the uncertain process $X_t = \exp(et, C_t)$ is called a geometric canonical process.

An assumption that the stock price follows geometric canonical process was presented by Liu [10]. In Liu’s stock model, the bond price $X_t$ and the stock price $Y_t$ are determined by

$$\begin{cases}
\frac{dX_t}{X_t} = r X_t dt \\
\frac{dY_t}{Y_t} = \epsilon X_t dt + \sigma X_t dC_t
\end{cases}$$

where $r$ is the riskless interest rate, $\epsilon$ is the stock drift, $\sigma$ is the stock diffusion, and $C_t$ is a canonical process.

Option pricing problem is a fundamental problem in financial market. European option are the most classic and useful option. A European call option is a contract that gives the holder the right to buy a stock at an expiration time $T$ for a strike price $K$. Liu [10] proposed the European option pricing formulae for Liu’s stock model.

### 3 American Call Option Price

An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time $T$ for a strike price $K$. Consider Liu’s stock model, we assume that an American call option has strike price $K$ and expiration time $T$. If $Y_t$ is the price of the underlying stock, then it is clear that the payoff from an American call option is the supremum of $(Y_t - K)^+$ over the time interval $[0, T]$, i.e.,

$$\sup_{0 \leq t \leq T} \exp(-rt)(Y_t - K)^+. \quad (2)$$

Hence the American call option price should be the expected present value of the payoff. Then this option has price

$$f_c = E\left[\sup_{0 \leq t \leq T} \exp(-rt)(Y_t - K)^+\right]. \quad (3)$$

In order to get this American call option price of Liu’s stock model, we need to solve the equation (3) in which $Y_t = Y_0 \exp(et + \sigma C_t)$. Before doing this, we will firstly calculate the distribution function $\Psi(x)$ of

$$\sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+.$$

For each $t \in (0, T]$, it is obvious that $\Phi_t(x) = 0$ when $x \leq 0$. If $x > 0$, we have $\Phi_t(x)$

$$= M \left\{ \exp(-rt) \left( Y_0 \exp(et + \sigma C_t) - K \right)^+ \leq x \right\}$$

$$= M \left\{ Y_0 \exp(et + \sigma C_t) \leq K + x \exp(rt) \right\}$$

$$= M \left\{ C_t \leq \frac{1}{\sigma} \ln \left( \frac{K + x \exp(rt)}{Y_0} \right) - \frac{et}{\sigma} \right\}$$

$$= \left(1 + \exp\left(\frac{\epsilon}{\sqrt{3\sigma t}} + \frac{\pi}{\sqrt{3\sigma t}} \ln \frac{Y_0}{K + x \exp(rt)}\right)\right)^{-1}. \quad (1)$$
In order to calculate the distribution function of 
\[ \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+, \]
we will use the extreme value theorem.

It is obvious that \( \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+ \) is an increasing function of independent increment process \( et + \sigma C_t \) and the distribution function \( \Phi_t(x) \) is continuous for each fixed \( t \in (0, T] \). By the Extreme Value Theorem 2, the distribution \( \Psi(x) \) is
\[
\Psi(x) = \inf_{0 \leq t \leq T} \Phi_t(x) \\
= \inf_{0 \leq t \leq T} \left( 1 + \exp\left( \frac{e}{\sqrt{3} \sigma t} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K + \exp(\pi Y_0)} \right) \right)^{-1} \\
= \left( 1 + \exp\left( \frac{e}{\sqrt{3} \sigma t} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K + \exp(\pi Y_0)} \right) \right)^{-1}.
\]

**Theorem 4.** Assume an American call option for the Liu’s stock model (1) has a strike price \( K \) and an expiration time \( s \). Then the American call option price is
\[ f_c = \exp(-rT)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp\left( \frac{\pi (\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} dy. \]

**Proof:** By the definition of expected value of uncertain variable, we have
\[
f_c = E\left[ \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+ \right] \\
= \int_0^{+\infty} \mathcal{M}\left\{ \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+ \geq x \right\} dx \\
= \int_0^{+\infty} (1 - \Psi(x)) dx \\
= \int_0^{+\infty} \left( 1 - \left( 1 + \exp\left( \frac{e}{\sqrt{3} \sigma T} + \frac{\pi}{\sqrt{3} \sigma T} \ln \frac{Y_0}{K + \exp(\pi Y_0)} \right) \right)^{-1} \right) dx \\
= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp\left( \frac{\pi (\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} dy.
\]

**Theorem 5.** American call option formula of Liu’s stock model (1) \( f_c = f(Y_0, K, e, \sigma, r, T) \) has the following properties:
(i) \( f \) is an increasing and convex function of \( Y_0 \);
(ii) \( f \) is a decreasing and convex function of \( K \);
(iii) \( f \) is an increasing function of \( e \);
(iv) \( f \) is an increasing function of \( \sigma \);
(v) \( f \) is an increasing function of \( T \);
(vi) \( f \) is a decreasing function of \( r \).

**Proof:** (i) If the other parameters are unchanged, the function \( \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 X - K)^+ \) is an increasing and convex function of \( Y_0 \) where \( X \) is any nonnegative constant. Thus the quantity \( \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+ \) is increasing and convex function of \( Y_0 \) and the uncertainty distribution of \( \exp(et + \sigma C_t) \) is independent of \( Y_0 \), therefore \( f \) is increasing and convex function of \( Y_0 \).

(ii) This is follows from the fact that \( \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 X - K)^+ \) is decreasing and convex of \( K \).

(iii) In the equation (3), it is obvious that \( \left( 1 + \exp\left( \frac{\pi (\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} \) is increasing function of \( e \). It means that \( f \) is increasing \( e \).

(iv) It is obvious that \( \left( 1 + \exp\left( \frac{\pi (\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} \) is increasing of \( \sigma \). Thus the European call price is increasing of \( \sigma \).

(v) It is easily to see that
\[ f_c = E\left[ \sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+ \right] \]
is increasing with \( T \).

(vi) Since \( \exp(-rt) \) is decreasing of \( r \), the European call price is decreasing of \( r \).

**Example 1.** Suppose that a stock is presently selling for a price of \( Y_0 = 40 \), the riskless interest rate \( r \) is 8% per annum, the stock drift \( e \) is 0.06 and the stock diffusion \( \sigma \) is 0.25. We would like to find an American put option price that expires in three months and has a strike price of \( K = 45 \).

**4 American Put Option Price**

An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time \( T \) for a strike price \( K \). Suppose that there is an American put option with strike price \( K \) and expiration \( T \) in Liu’s stock model. If \( Y_t \) is the price of the underlying stock, then it is clear that the payoff from an American put option is the supremum of \( (K - Y_t)^+ \) over the time interval \( [0, s] \), i.e.,
\[ \sup_{0 \leq t \leq T} (K - Y_t)^+ \]
Hence the American put option price should be the expected present value of the payoff.

**Definition 4.** Assume an American put option has a strike price \( K \) and an expiration time \( T \). Then this option has the price
\[ f_p = E\left[ \sup_{0 \leq t \leq T} \exp(-rt)(K - Y_t)^+ \right]. \tag{4} \]

In order to get this American option price of Liu’s stock model, we need to solve the equation (4) in which \( Y_t = Y_0 \exp(et + \sigma C_t) \). Before doing this, we will firstly calculate the distribution function \( \Psi(x) \) of
\[ \sup_{0 \leq t \leq T} \exp(-rt)(K - Y_t)^+ \].
For each $t \in (0, T]$ and $x < K \exp(-rt)$, the distribution function $\Phi_t(x)$ is

$$\Phi_t(x) = \mathcal{M}\left\{ \left. \exp(-rt)(K - Y_0 \exp(et + \sigma C_t) + x \right\}$$

$$= 1 - \mathcal{M}\{Y_0 \exp(et + \sigma C_t) < K - \exp(rt)\}$$

$$= \left(1 + \exp\left(-\frac{e}{\sqrt{3} \sigma} - \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1}.$$

In order to calculate the distribution function of $\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_t)^+$, we need to use extreme value theorem.

By the extreme value Theorem 3, the uncertainty distribution function $\Psi(x)$ of

$$\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_t)^+$$

is $\Psi(x) = 1 - \sup_{0 \leq t \leq T} \left(1 + \exp\left(\frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1}$.

**Theorem 6.** Assume an American put option for Liu’s stock model (1) has a strike price $K$ and an expiration time $s$. Then the American put option price is $f_p = \int_0^{K \exp(-rt)} \sup_{0 \leq t \leq T} \left(1 + \exp\left(\frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1} \, dx$.

**Proof:** It follows from the definition of expected value of uncertain variables that

$$f_p = E\left[\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_0 \exp(et + \sigma C_t) + x)\right]$$

$$= \int_0^{K \exp(-rt)} \mathcal{M}\{\sup_{0 \leq t \leq T} (K - Y_0 \exp(et + \sigma C_t)) + x \geq x\} \, dx$$

$$= \int_0^{K \exp(-rt)} (1 - \Psi(x)) \, dx$$

$$= \int_0^{K \exp(-rt)} \sup_{0 \leq t \leq T} \left(1 + \exp\left(\frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1} \, dx.$$

**Theorem 7.** American put option formula of Liu’s stock model (1) $f = f(Y_0, K, e, \sigma, r, T)$ has the following properties:

(i) $f$ is a decreasing and convex function of $Y_0$;
(ii) $f$ is an increasing and convex function of $K$;
(iii) $f$ is a decreasing function of $e$;
(iv) $f$ is an increasing function of $\sigma$;
(v) $f$ is a decreasing function of $r$;
(vi) $f$ is an increasing function of $T$.

**Proof:** (i) If the other parameters are unchanged, the function $\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_0 X)^+$ is a decreasing and convex function of $Y_0$ where $X$ is any nonnegative constant. Thus the quantity

$$\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_0 \exp(et + \sigma C_t) + x)^+$$

is a decreasing and convex function of $Y_0$ and the uncertainty distribution of $\exp(et + \sigma C_t)$ is independent of $Y_0$, therefore $f$ is decreasing and convex function of $Y_0$.

(ii) It follows the fact that $\sup_{0 \leq t \leq T} \exp(-rt)(K - Y_0 X)^+$ is a decreasing and convex function of $K$.

(iii) In the equation (3), it is obvious that $\sup_{0 \leq t \leq T} \left(1 + \exp\left(\frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1}$ is a decreasing function of $e$. It means that $f$ is decreasing.

(iv) It follows from that $\sup_{0 \leq t \leq T} \left(1 + \exp\left(\frac{e}{\sqrt{3} \sigma} + \frac{\pi}{\sqrt{3} \sigma t} \ln \frac{Y_0}{K - \exp(rt)}\right)\right)^{-1}$ is a decreasing function of $\sigma$. Thus the European call price is increasing of $\sigma$.

(v) Since $\exp(-rt)$ is decreasing of $r$, the European call price is decreasing of $r$.

(vi) It is easily to see that $f_c = E\left[\sup_{0 \leq t \leq T} \exp(-rt)(Y_0 \exp(et + \sigma C_t) - K)^+\right]$ is increasing with $T$.

**Example 2.** Suppose that a stock is presently selling for a price of $Y_0 = 40$, the riskless interest rate $r$ is 8% per annum, the stock drift $e$ is 0.06 and the stock diffusion $\sigma$ is 0.25. We would like to find an American put option price that expires in three months and has a strike price of $K = 35$.

## 5 Conclusion

In this paper, we investigated the option pricing problems for uncertain financial market. American call and put option price formulas were computed for Liu’s stock model. Some mathematical properties of these formulas were studied.

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**References**


