Uncertain Optimal Control with Jump

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Abstract

Based on the uncertainty theory, an uncertain optimal control problem with jump is considered for uncertain dynamical systems driven by both an uncertain $V$ jump process and an uncertain canonical process. The principle of optimality and the equation of optimality are obtained by applying the dynamic programming principle of the optimal control. As its applications, a pension funds control problem is discussed and the optimal strategies are presented.

\textbf{Keywords:} optimal control, uncertainty, $V$ jump process, equation of optimality, pension funds

1 Introduction

Optimal control theory is an important branch of modern control theory. It is to seek the optimal control decision among the admissible control strategies for maximizing or minimizing some objective functions, which relate to a dynamic process driven by a differential equation. With the more use of methods and results on mathematics and computer science, optimal control theory has made considerable advances, and has been used widely in real-life problems such as production engineering, national defence, programming, finance, and economic management.

Since Merton [22] studied a stochastic optimal control problem for finance in seventies of the last century, stochastic optimal control problem has been investigated by many researchers. Fleming and Rishel [9], Harrison [11], Karatzas [13], and Stengel [25] dealt with

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The concept of fuzzy optimal control was presented by Komolov et al. [15] in 1979. Since then, many researchers studied the fuzzy optimal control problem. Filev and Angelov [8] studied the fuzzy optimal control problem of nonlinear systems on the basis of fuzzy mathematical programming. Jia and Zhang [12] proposed the fuzzy multiobjective optimal control model. Ostermark [23] analyzed application of the fuzzy optimal control model to dynamic portfolio management. These work concerned on finding a fuzzy control to optimize an objective subject to a dynamic system.

Recently, based on the credibility theory found by Liu [16] in 2004, Zhu [29] proposed and dealt with a fuzzy optimal control problem with fuzzy process by using dynamic programming. Wang and Zhu [27] further studied multidimensional fuzzy optimal control problem and its application to optimal stopping time. Zhao and Zhu [28] discussed linear quadratic fuzzy optimal control problem. Zhu [31] investigated a fuzzy optimal control problem for multi-stage fuzzy systems. These work, which is different from the work of Komolov, is to find a control to optimize an objective subject to a fuzzy differential equation driven by a fuzzy process.

All the above research work on optimal control problems assumes that control system is under random or fuzzy environment, and tools of studying these types of optimal control problem are probability theory, fuzzy set theory or credibility theory. However, the complexity of the world makes the events we face uncertain in various forms. A lot of surveys showed that in many cases, the uncertainty behaves neither like randomness nor like fuzziness. In the real life, some information and knowledge, which usually are represented by human language like “about 100km”, “approximately 39°C”, “roughly 80kg”, “low speed”, “middle age”, and “big size”, behave neither like randomness nor like fuzziness. In order to deal with this type of uncertainty, an uncertainty theory was invented by Liu [17] in 2007 and refined in 2010 [21] based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Now the content of uncertainty theory has been developed to a fairly complete system and has gained quite a high achievement both in theoretical aspect and practical aspect.

In practice, numerous uncertain optimal control problems with this type of uncertainty of human system need to be solved. Developing the uncertain optimal control theory and method systemically not only has profound theoretical value but also has broad prospects for applications. Based on uncertain canonical process in uncertainty theory, Zhu [30] introduced and dealt with an uncertain optimal control problem by using dynamic programming in 2010. It is reasonable model for uncertain systems of following the continuous uncertain canonical process without jump. Nevertheless, in real world, there are also many uncertain systems whose state processes follow uncertain processes.
with jump. For example, in many cases the stock price may jump at scheduled or unscheduled times because of economic crisis, war, announcements of economic statistics, announcements of monetary policy, and so on. These factors should be incorporated into stock price model. Hence, in this paper, we will present and deal with an uncertain optimal control problem with jump by considering the effects of jumps on the optimal policies. It is an extension of the model proposed by Zhu [30].

The remainder of the paper is structured as follows. In next section, some necessary elementary concepts and theorems about uncertain theory will be recalled. In Section 3, a $Z$ jump uncertain variable and a $V$ jump process will be introduced and discussed. After that, an uncertain optimal control problem with jump will be proposed, and the principle of optimality be derived by Bellman’s dynamic programming principle in Section 4. Then, in Section 5, the equation of optimality for uncertain optimal control problem with jump will be obtained. As its applications, in Section 6, a pension funds control problem will be discussed and the optimal strategies will be gained.

2 Preliminary

For the better understanding of this paper, let us first review some concepts of uncertainty theory, for example, uncertain measure, uncertainty space, uncertainty distribution, expected value of uncertain variable, uncertain process, canonical process, uncertain differential equation, and some theorems.

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. For any $\Lambda \in \mathcal{L}$, Liu [17] introduced an uncertain measure $\mathcal{M}(\Lambda)$ to express the chance that uncertain event $\Lambda$ occurs. If the set function $\mathcal{M}\{\cdot\}$ on $\mathcal{L}$ satisfies that $\mathcal{M}\{\Gamma\} = 1; \mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$ for $\Lambda_1, \Lambda_2 \in \mathcal{L}; \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}; \mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\{\Lambda_i\}$. Then $\mathcal{M}$ is called an uncertain measure, and the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. Moreover, an uncertain variable is defined as a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

The uncertainty distribution $\Phi: \mathbb{R} \to [0, 1]$ of an uncertain variable $\xi$ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. The expected value of an uncertain variable $\xi$ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathcal{M}\{\xi \geq r\} \, dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\} \, dr$$

provided that at least one of the two integrals is finite. The variance of $\xi$ is $V[\xi] = E[(\xi - E[\xi])^2]$.

Liu [19] introduced the independence concept of uncertain variables. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are independent if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}.$$ 

for any Borel set $B_1, B_2, \ldots, B_m$ of real numbers.
Theorem 2.1 (Liu [19]) Let $\xi$ and $\eta$ be independent uncertain variables with finite expected values. Then for any numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

Based on the uncertainty space, Liu [19] introduced the concepts of uncertain process, independent increments process, stationary increments process, canonical process, etc. An uncertain process $X_t$ is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. An uncertain process $X_t$ is an independent increments process if and only if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables for any times $t_0 < t_1 < \cdots < t_k$. An uncertain process $X_t$ is a stationary increments process if and only if, for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$. A canonical process $C_t$ is an uncertain process if and only if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous, (ii) $C_t$ has stationary and independent increments, (iii) every increment $C_{s+t} - C_s$ is a normally distributed uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$  

Based on canonical process, an uncertain differential equation is defined as

$$dX_t = g_1(X_t, t)dt + g_2(X_t, t)dC_t$$  \hspace{1cm} (2.1)

where $C_t$ is a canonical process, and $g_1$ and $g_2$ are some given functions. A solution is an uncertain process $X_t$ that satisfies (2.1) for almost all $t$.

The following results about uncertain variables are useful in the discussion of the paper.

Theorem 2.2 (Zhu [30]) Let $\xi$ be a normally distributed uncertain variable with the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$  

Then for any real number $a$,

$$\frac{\sigma^2}{2} \leq E[a\xi + \xi^2] \leq \sigma^2.$$  

Theorem 2.3 (Liu [21]) Let $f$ be a convex function on $[a, b]$, and $\xi$ an uncertain variable that takes values in $[a, b]$ and has expected value $e$. Then

$$E[f(\xi)] \leq \frac{b - e}{b - a} f(a) + \frac{e - a}{b - a} f(b).$$  \hspace{1cm} (2.2)
3 Jump Uncertain Process

The importance of jumps is apparent for uncertain optimal control problems. Some external extreme events or noises have a great influence on uncertain dynamic systems. For instance, at important events or macroeconomic announcements, there can be large changes in the value of financial portfolios. In order to deal with an uncertain optimal control problem with jump, the discontinuous jump processes must be modeled into uncertain differential equation which describes uncertain dynamic systems. In stochastic dynamic systems, a jump is generally modeled as a Poisson process. In this paper, we model the discontinuous jump part of an uncertain system by introducing a new so-called $V$ jump process, which is associated with a so-called $Z$ jump uncertain variable $Z(r_1, r_2, t)$ defined by a jump uncertainty distribution. To simplify analysis, we assume that there is only one jump point. For multi-jump point cases, we may do analogous analysis. Many empirical investigations and analysis show that this type of jump process has real background such as the discontinuous stock price change process in uncertain financial market. In addition, this $V$ jump process is suitable for analyzing the uncertain optimal control model studied in the paper. Of course, in real life, there may be some other types of uncertain jump processes.

3.1 Jump uncertain variable

To begin with, we introduce a jump uncertain variable.

**Definition 3.1** An uncertain variable $Z(r_1, r_2, t)$ is said to be a $Z$ jump uncertain variable with parameters $r_1$ and $r_2$ ($0 < r_1 < r_2 < 1$) for $t > 0$ if it has a jump uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{2r_1}{t} x, & \text{if } 0 \leq x < \frac{t}{2} \\
r_2 + \frac{2(1-r_2)}{t} \left(x - \frac{t}{2}\right), & \text{if } \frac{t}{2} \leq x < t \\
1, & \text{if } x \geq t
\end{cases}
$$

(3.1)

The uncertain distribution $\Phi$ of a $Z$ jump uncertain variable has a discontinuous point at which the value of $\Phi$ has a jump with step $r_2 - r_1$ as Fig.1.

**Theorem 3.1** Assume $\xi_1$ and $\xi_2$ are independent $Z$ jump uncertain variables $Z(r_1, r_2, t_1)$ and $Z(r_1, r_2, t_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a $Z$ jump uncertain variable $Z(r_1, r_2, t_1 + t_2)$, i.e.,

$$
Z(r_1, r_2, t_1) + Z(r_1, r_2, t_2) = Z(r_1, r_2, t_1 + t_2)
$$

(3.2)

The product of a $Z$ jump uncertain variable $Z(r_1, r_2, t)$ and a scalar number $k > 0$ is also a $Z$ jump uncertain variable $Z(r_1, r_2, kt)$, i.e.,

$$
k \cdot Z(r_1, r_2, t) = Z(r_1, r_2, kt)
$$

(3.3)
Figure 1: Distribution of $Z(r_1, r_2, t)$

**Proof:** Assume that the uncertain variables $\xi_1$ and $\xi_2$ have uncertainty distributions $\Phi_1(x)$ and $\Phi_2(x)$, respectively. That is, for $i = 1, 2$,

$$
\Phi_i(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{2r_1}{t_i} x, & \text{if } 0 \leq x < \frac{t_i}{2} \\
r_2 + \frac{2(1 - r_2)}{t_i} \left( x - \frac{t_i}{2} \right), & \text{if } \frac{t_i}{2} \leq x < t_i \\
1, & \text{if } x \geq t_i 
\end{cases}
$$

Then for $\alpha \in (0, r_1) \cup [r_2, 1)$, we have

$$
\Phi_i^{-1}(\alpha) = \begin{cases} 
\frac{t_i}{2r_1} \alpha, & \text{if } 0 < \alpha < r_1 \\
\frac{(\alpha - r_2)t_i}{2(1 - r_2)} + \frac{t_i}{2}, & \text{if } r_2 \leq \alpha < 1 
\end{cases}
$$

for $i = 1, 2$. Thus for $\alpha \in (0, r_1) \cup [r_2, 1)$, we have

$$
\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = \begin{cases} 
\frac{t_1 + t_2}{2r_1} \alpha, & \text{if } 0 < \alpha < r_1 \\
\frac{(\alpha - r_2)(t_1 + t_2)}{2(1 - r_2)} + \frac{t_1 + t_2}{2}, & \text{if } r_2 \leq \alpha < 1 
\end{cases}
$$

Let $\Phi : \mathbb{R} \to [0, 1]$ be an increasing function satisfying that $\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha)$ for $\alpha \in (0, r_1) \cup [r_2, 1)$. Then for any $x \in (0, t_1 + t_2)$, there is an $\alpha \in (0, r_1) \cup [r_2, 1)$ such that $x = \Phi^{-1}(\alpha)$. On one hand,

$$
\mathcal{M}\{\xi_1 + \xi_2 \leq \Phi^{-1}(\alpha)\} \\
= \mathcal{M}\{\xi_1 + \xi_2 \leq \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha)\} \\
\geq \mathcal{M}\{(\xi_1 \leq \Phi_1^{-1}(\alpha)) \cap (\xi_2 \leq \Phi_2^{-1}(\alpha))\} \\
= \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \land \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\} \\
= \alpha \land \alpha = \alpha.
$$
On the other hand, there exists an index \( i = 1 \) or \( i = 2 \) such that

\[
\{\xi_1 + \xi_2 \leq \Phi^{-1}_1(\alpha) + \Phi^{-1}_2(\alpha)\} \subset \{\xi_i \leq \Phi^{-1}_i(\alpha)\}.
\]

Thus

\[
\mathcal{M}\{\xi_1 + \xi_2 \leq \Phi^{-1}(\alpha)\} \leq \mathcal{M}\{\xi_i \leq \Phi^{-1}(\alpha)\} = \alpha.
\]

It follows that \( \mathcal{M}\{\xi_1 + \xi_2 \leq x\} = \mathcal{M}\{\xi_1 + \xi_2 \leq \Phi^{-1}(\alpha)\} = \alpha \). In other words, \( \Phi \) is just the uncertainty distribution of \( \xi_1 + \xi_2 \) provided by

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{2r_1}{t_1 + t_2}x, & \text{if } 0 \leq x < \frac{t_1 + t_2}{2} \\
r_2 + \frac{2(1 - r_2)}{t_1 + t_2}\left(x - \frac{t_1 + t_2}{2}\right), & \text{if } \frac{t_1 + t_2}{2} \leq x < t_1 + t_2 \\
1, & \text{if } x \geq t_1 + t_2
\end{cases}
\]

Hence the sum \( \xi_1 + \xi_2 \) is also a \( Z \) jump uncertain variables \( Z(r_1, r_2, t_1 + t_2) \). The first part is verified.

Next, denote the uncertainty distribution of a \( Z \) jump uncertain variable \( \xi \sim Z(r_1, r_2, t) \) by \( \Phi_\xi \). Then for \( k > 0 \), we have

\[
\Phi_{k\xi}(x) = \mathcal{M}\{k\xi \leq x\} = \mathcal{M}\{\xi \leq x/k\} = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{2r_1}{kt}x, & \text{if } 0 \leq x < \frac{kt}{2} \\
r_2 + \frac{2(1 - r_2)}{kt}\left(x - \frac{kt}{2}\right), & \text{if } \frac{kt}{2} \leq x < kt \\
1, & \text{if } x \geq kt
\end{cases}
\]

Hence \( k\xi \) is just a \( Z \) jump uncertain variable \( Z(r_1, r_2, kt) \). The theorem is proved.

For multi-jump point case, we may have similar definition and property.

**Definition 3.2** An uncertain variable \( Z(r_{i1}, r_{i2}, n, t) \) is said to be a \( Z \) jump uncertain variable with parameters \( r_{i1} \) and \( r_{i2} \) \( (0 < r_{i1} < r_{i2} < r_{(i+1)1} < r_{(i+1)2} < 1, \ i = 1, 2, \ldots, n) \) for \( t > 0 \) if it has a jump uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{r_{i1}}{t_1}x, & \text{if } 0 \leq x < t_1 \\
r_{i2} + \frac{r_{(i+1)1} - r_{i2}}{t_{i+1} - t_i}(x - t_i), & \text{if } t_i \leq x < t_{i+1} \ (i = 1, 2, \ldots, n) \\
1, & \text{if } x \geq t_{n+1} = t
\end{cases}
\]
Theorem 3.2 Assume $\xi_1$ and $\xi_2$ are independent $Z$ jump uncertain variables $Z(r_{i1}, r_{i2}, n, t)$ and $Z(r_{i1}, r_{i2}, n, s)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a $Z$ jump uncertain variable $Z(r_{i1}, r_{i2}, n, t + s)$, i.e.,

$$Z(r_{i1}, r_{i2}, n, t) + Z(r_{i1}, r_{i2}, n, s) = Z(r_{i1}, r_{i2}, n, t + s) \quad (3.5)$$

The product of a $Z$ jump uncertain variable $Z(r_{i1}, r_{i2}, n, t)$ and a scalar number $k > 0$ is also a $Z$ jump uncertain variable $Z(r_{i1}, r_{i2}, n, kt)$, i.e.,

$$k \cdot Z(r_{i1}, r_{i2}, n, t) = Z(r_{i1}, r_{i2}, n, kt) \quad (3.6)$$

The proof of the theorem is similar to theorem 3.1.

3.2 Jump uncertain process

Now we will define a jump uncertain process by a $Z$ jump uncertain variable.

Definition 3.3 An uncertain process $V_t$ is said to be a $V$ jump process with parameters $r_1$ and $r_2$ ($0 < r_1 < r_2 < 1$) for $t \geq 0$ if (i) $V_0 = 0$, (ii) $V_t$ has stationary and independent increments, (iii) every increment $V_s + t - V_s$ is a $Z$ jump uncertain variable $Z(r_{i1}, r_{i2}, n, t)$.

Let $V_t$ be a $V$ jump uncertain process, and $\Delta V_t = V_{t+\Delta t} - V_t$. Then

$$E[\Delta V_t] = \int_0^\Delta (1 - \Phi(x))dx$$

$$= \int_0^\Delta (1 - \Phi(x))dx$$

$$= \int_0^{\Delta t} \left( 1 - \frac{2r_1 x}{\Delta t} \right)dx + \int_0^{\Delta t} \left( 1 - r_2 - \frac{2(1-r_2)}{\Delta t} \left( x - \Delta t / 2 \right) \right)dx$$

$$= \frac{3 - r_1 - r_2}{4} \Delta t.$$

Theorem 3.3 (Existence Theorem) There is a $V$ jump uncertain process.

Proof: Without loss of generality, we only prove that there is a $V$ jump uncertain process on the range of $t \in [0, 1]$. Let

$$\{\xi(r) \mid r \text{ represents rational numbers in } [0, 1]\}$$

be a countable sequence of independent $Z$ jump uncertain variables $Z(r_1, r_2, 1)$. For each integer $n$, we define an uncertain process: $V_n(0) = 0$, and

$$V_n(t) = \begin{cases} \frac{1}{n} \sum_{i=1}^k \xi \left( \frac{i}{n} \right), & \text{if } t = \frac{k}{n} \quad (k = 1, 2, \cdots, n) \\ \text{linear}, & \text{otherwise.} \end{cases}$$
For $t = k/n$, by Theorem 3.1, we know that

$$V_n(t) = \frac{1}{n} \sum_{i=1}^{k} \xi \left( \frac{i}{n} \right)$$

is also a $Z$ jump uncertain variable $Z(r_1, r_2, k/n)$, namely, $Z(r_1, r_2, t)$. Hence the limit

$$\lim_{n \to \infty} V_n(t)$$

exists almost surely, and

$$V(t) = \lim_{n \to \infty} V_n(t) = Z(r_1, r_2, t).$$

We may verify that the limit $V(t)$ meets the conditions of $V$ jump process. In fact, for $t_1 = k_1/n$ and $t_2 = k_2/n$ ($k_1 < k_2$), we have

$$V_n(t_2) - V_n(t_1) = \frac{1}{n} \sum_{i=k_1+1}^{k_2} \xi \left( \frac{i}{n} \right) = Z(r_1, r_2, \frac{k_2 - k_1}{n}) \sim Z(r_1, r_2, t_2 - t_1).$$

Then

$$V(t_2) - V(t_1) = \lim_{n \to \infty} [V_n(t_2) - V_n(t_1)] \sim Z(r_1, r_2, t_2 - t_1).$$

Thus $V(t)$ has stationary increment. On the other hand, for $t_1 = k_1/n < t_2 = k_2/n < t_3 = k_3/n < t_4 = k_4/n$, we have

$$V_n(t_2) - V_n(t_1) = \frac{1}{n} \sum_{i=k_1+1}^{k_2} \xi \left( \frac{i}{n} \right), \quad V_n(t_4) - V_n(t_3) = \frac{1}{n} \sum_{i=k_3+1}^{k_4} \xi \left( \frac{i}{n} \right).$$

We know that $V_n(t_2) - V_n(t_1)$ is independent of $V_n(t_4) - V_n(t_3)$. Therefore, $V(t_2) - V(t_1)$ is independent of $V(t_4) - V(t_3)$. This means that $V(t)$ has independent increment. Hence there is a $V$ jump uncertain process.

**Theorem 3.4** Let $C_t$ be an uncertain canonical process, and $V_t$ a $V$ jump uncertain process. Denote $\zeta = b\xi + d\eta$, where $\xi = \Delta C_t$, $\eta = \Delta V_t$, $b, d \in \mathbb{R}$. Let $\xi$ and $\eta$ are independent. Then for any real number $a$,

$$E[a\zeta + \zeta^2] = \frac{ad(3 - r_1 - r_2)}{4} \Delta t + o(\Delta t).$$

**Proof:** To begin with we have

$$E[a\zeta + \zeta^2] \geq E[a\zeta] = aE[b\xi + d\eta] = a(bE[\xi] + dE[\eta]) = \frac{ad(3 - r_1 - r_2)}{4} \Delta t. \quad (3.7)$$

On the other hand, since

$$a\zeta + \zeta^2 = a(b\xi + d\eta) + b^2\xi^2 + d^2\eta^2 + 2bd\xi\eta$$

$$\leq a(b\xi + d\eta) + 2(b^2\xi^2 + d^2\eta^2)$$

$$= (ab\xi + 2b^2\xi^2) + (ad\eta + 2d^2\eta^2),$$
and $ab\xi + 2b^2\xi^2$ and $ad\eta + 2d^2\eta^2$ are independent, we have

$$E[a\zeta + \zeta^2] \leq E[ab\xi + 2b^2\xi^2] + E[ad\eta + 2d^2\eta^2].$$

It follows from Theorem 2.2 that $E[ab\xi + 2b^2\xi^2] = o(\Delta t)$. Hence we can get

$$E[ad\eta + 2d^2\eta^2] \leq \frac{E[\eta]}{\Delta t} (ad\Delta t + 2d^2(\Delta t)^2)$$

$$= \frac{3 - r_1 - r_2}{4} (ad\Delta t + 2d^2(\Delta t)^2)$$

$$= \frac{ad(3 - r_1 - r_2)}{4} \Delta t + o(\Delta t)$$

by Theorem 2.3. Thus

$$E[a\zeta + \zeta^2] \leq \frac{ad(3 - r_1 - r_2)}{4} \Delta t + o(\Delta t).$$

Combining inequality (3.7) and inequality (3.8), we can obtain

$$E[a\zeta + \zeta^2] = \frac{ad(3 - r_1 - r_2)}{4} \Delta t + o(\Delta t).$$

### 4 Uncertain Optimal Control Problem with Jump

Uncertain optimal control problem with jump is to seek the optimal control decision or optimal control law in every possible control strategies for optimizing some objective functions, which relate to an uncertain process with jump provided by an uncertain differential equation with jump. Since it may not be possible to optimize the objective functions as real functions because the objective functions are uncertain variables for any decision, we should know how to compare two different uncertain variables, or how to decide which is larger. Actually, there are many criteria to do so, such as, expected value, optimistic value, pessimistic value, and uncertain measure [20]. In this paper we adopt the expected value-based method as in [30].

Unless in particular stated otherwise, we assume $C_t$ is a canonical process and $V_t$ a $V$ jump uncertain process with parameters $r_1$ and $r_2$ $(0 < r_1 < r_2 < 1)$, and $\Delta C_t$ and $\Delta V_t$ are independent. Now we present an uncertain expected value optimal control problem with jump as follows

$$\begin{cases}
J(0, x_0) \equiv \sup_{u_s} \mathbb{E} \left[ \int_0^T f(s, X_s, u_s)ds + G(T, X_T) \right] \\
\text{subject to} \quad dX_s = \nu(s, X_s, u_s)ds + \gamma(s, X_s, u_s)dC_s + \chi(s, X_s, u_s)dV_s \\
X_0 = x_0
\end{cases}$$

(4.1)

(4.2)

where $X_s$ denotes the state variable, $u_s$ the decision (control) variable (represents the function $u(s, X_s)$ of time $s$ and state $X_s$), $f$ the objective function, and $G$ the function of
terminal reward. The symbol $E$ represents the expected value operator of an uncertain variable, and $J(0, x_0)$ is the expected optimal reward obtainable in $[0, T]$ with the initial condition that at time 0 we are in state $x_0$. For a given $u_s$, $dX_s$ is governed by the uncertain differential equation (4.2), where $\nu$, $\gamma$ and $\chi$ are three functions of time $s$, state $X_s$ and control $u_s$.

For any $0 < t < T$, let $J(t, x)$ denote the expected optimal reward obtainable in $[t, T]$ with the condition that at time $t$ we are in state $X_t = x$. Then we have

\[
J(t, x) = \sup_{u_s} E \left[ \int_t^T f(s, X_s, u_s)ds + G(T, X_T) \right]
\]

subject to

\[
dX_s = \nu(s, X_s, u_s)ds + \gamma(s, X_s, u_s)dC_s + \chi(s, X_s, u_s)dV_s
\]

$X_t = x$.

\[ (4.3) \]

Now we give the following principle of optimality for uncertain optimal control with jump.

**Theorem 4.1 (Principle of optimality)** For any $(t, x) \in [0, T) \times \mathbb{R}$, and $\Delta t > 0$ with $t + \Delta t < T$, we have

\[
J(t, x) = \sup_{u_s} E \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + J(t + \Delta t, x + \Delta X_t) \right],
\]

where $x + \Delta X_t = X_{t+\Delta t}$.

The proof of the theorem is parallel to the corresponding result in Zhu [30].

## 5 Equation of Optimality

Next we give a fundamental result called equation of optimality of the uncertain optimal control problem with jump (4.3).

**Theorem 5.1 (Equation of optimality)** Let $J(t, x)$ be twice differentiable on $[0, T] \times \mathbb{R}$. Then we have

\[
-J(t, x) = \sup_{u_t} \left\{ f(t, x, u_t) + J_x(t, x)\nu(t, x, u_t) + \frac{3 - r_1 - r_2}{4} \chi(t, x, u_t)J_x(t, x) \right\}
\]

where $J_t(t, x)$ and $J_x(t, x)$ are the partial derivatives of the function $J(t, x)$ in $t$ and $x$, respectively.

**Proof:** For any $\Delta t > 0$, we have

\[
\int_t^{t+\Delta t} f(s, X_s, u_s)ds = f(t, x, u(t, x))\Delta t + o(\Delta t)
\]

\[ (5.2) \]
By using Taylor series expansion, we get
\[ J(t + \Delta t, x + \Delta X_t) = J(t, x) + J_t(t, x)\Delta t + J_x(t, x)\Delta X_t + \frac{1}{2} J_{tt}(t, x)\Delta^2 t \]
\[ + \frac{1}{2} J_{xx}(t, x)\Delta X_t^2 + J_{tx}(t, x)\Delta t\Delta X_t + o(\Delta t) \]  \hspace{1cm} (5.3)

Substituting Equations (5.2) and (5.3) into Equation (4.4) yields
\[ 0 = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + J_t(t, x)\Delta t + E\left[ J_x(t, x)\Delta X_t + \frac{1}{2} J_{tt}(t, x)\Delta^2 t \right] \right. \]
\[ + \left. \frac{1}{2} J_{xx}(t, x)\Delta X_t^2 + J_{tx}(t, x)\Delta t\Delta X_t \right\} + o(\Delta t) \]  \hspace{1cm} (5.4)

Let \( \eta \) be an uncertain variable such that \( \Delta X_t = \eta + \nu(t, x, u_t)\Delta t \). It follows from (5.4) that
\[ 0 = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + J_t(t, x)\Delta t + J_x(t, x)\nu(t, x, u_t)\Delta t + E\left[ (J_x(t, x) \right. \right. \]
\[ + \left. \left. J_{xx}(t, x)\nu(t, x, u_t)\Delta t + J_{tx}(t, x)\Delta t)\eta + \frac{1}{2} J_{xx}(t, x)\eta^2 \right] + o(\Delta t) \right\} \]
\[ = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + J_t(t, x)\Delta t + J_x(t, x)\nu(t, x, u_t)\Delta t \right. \]
\[ + \left. E[a\eta + b\eta^2] + o(\Delta t) \right\} \]  \hspace{1cm} (5.5)

where \( a \equiv J_x(t, x) + J_{xx}(t, x)\nu(t, x, u_t)\Delta t + J_{tx}(t, x)\Delta t \), and \( b \equiv \frac{1}{2} J_{xx}(t, x) \). It follows from the uncertain differential equation, the constraint in (4.3), that \( \eta = \Delta X_t - \nu(t, x, u_t)\Delta t = \gamma(t, x, u_t)\Delta C_t + \chi(t, x, u_t)\Delta V_t \) is an uncertain variable. Theorem 3.4 implies that
\[ E[a\eta + b\eta^2] = \frac{a(3 - r_1 - r_2)}{4} \chi(t, x, u_t)\Delta t + o(\Delta t) \]
\[ = \frac{3 - r_1 - r_2}{4} \chi(t, x, u_t)J_x(t, x)\Delta t + o(\Delta t) \]  \hspace{1cm} (5.6)

Substituting Equation (5.6) into Equation (5.5) yields
\[ -J_t(t, x)\Delta t = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + J_x(t, x)\nu(t, x, u_t)\Delta t \right. \]
\[ + \left. \frac{3 - r_1 - r_2}{4} \chi(t, x, u_t)J_x(t, x)\Delta t + o(\Delta t) \right\} \]  \hspace{1cm} (5.7)

Dividing Equation (5.7) by \( \Delta t \), and letting \( \Delta t \to 0 \), we can obtain the result (5.1). The theorem is proved.

**Remark 5.1** The equation of optimality (5.1) gives a necessary condition for an extremum. When the equation has solutions, we may derive the optimal control strategy and optimal expected value of objective function. When \( f \) is concave in its arguments, then the equation will produce a maximum, and when function \( f \) is convex in its arguments, then it will produce a minimum. In addition, we note that the boundary condition of the equation is \( J(T, X_T) = G(T, X_T) \).
6 Optimal Control of Pension Funds

The analysis and control of pension fund dynamics is becoming increasingly important because of its upmost importance for the social and political stability of the wealthy economies. The dynamic control-theoretical framework was first applied to a pension funding problem by Boulier et al. [2]. Cairns [4], Cairns and Parker [5] studied discrete-time models. Boulier et al. [3] and Cairns [6] discussed the continuous-time pension fund modeling. In their researches, the target of pension fund control is trying to choose the optimal contribution rate and the asset allocation proportion for minimizing a value function which discounts exponentially future random values of a quadratic loss function by assuming that the pension fund can be invested in a risk-free asset and a risky asset whose return follows random jump-diffusion processes. If we assume that the return of a risk asset follows an uncertain jump process, along with uncertain canonical process, then pension funds control problem may be solved by uncertain optimal control with jump.

We assume that the pension fund dynamics is governed by the uncertain differential equation:

\[
\frac{dX_t}{X_t} = \frac{u_t - B}{X_t} dt + \frac{\sigma_1 w X_t}{X_t} dC_t + \frac{\sigma_2 w X_t}{X_t} dV_t.
\]

In this equation, \(X_t\) is the fund amount at time \(t\), \(u_t\) is the contribution rate at time \(t\) which is a predictable process and provides us with one of the means of controlling the dynamics of the pension fund, \(B\) is the pension scheme benefit outgo, while \(dR_t\) is the instantaneous return on assets in the interval \((t, t+dt)\). The pension scheme benefit outgo is assumed to be constant and deterministic.

Assume that the fund is to be invested in a risk-free asset and a risky asset. Namely, assume that the return rate on the risk-free asset is \(b\), and the return rate on the risky asset follows a jump uncertain process: \(dR_t = \mu dt + \sigma_1 dC_t + \sigma_2 dV_t\). Also assume that a proportion \(w\) of the fund is to be allocated in the risky asset whilst \(1-w\) is to be allocated in the risk-free asset. Thus the instantaneous return \(dR_t\) is then given by

\[
dR_t = b(1-w)dt + w(\mu dt + \sigma_1 dC_t + \sigma_2 dV_t).
\]

Hence the fund dynamics can be written as

\[
dX_t = \{[b + (\mu - b)w]X_t + u_t - B\}dt + \sigma_1 w X_t dC_t + \sigma_2 w X_t dV_t.
\]

Following Cairns [6] we assume that the goal of the funds is to choose the optimal contribution and asset allocation policies for minimizing a value function which discounts exponentially future uncertain values of a quadratic loss function over an infinite time horizon. Thus we consider the following optimal control problem of pension funds:

\[
\begin{align*}
J(t, x) \equiv & \min_{u_t, w} \mathbb{E} \left[ \int_t^{\infty} e^{-\beta s} \left\{ \alpha_1 (u_s - c_m)^2 + \alpha_2 (w X_s - x_p)^2 \right\} ds \right] \\
\text{subject to} \\
& \frac{dX_t}{X_t} = \{[b + (\mu - b)w]X_t + u_t - B\}dt + \sigma_1 w X_t dC_t + \sigma_2 w X_t dV_t,
\end{align*}
\]

where \(\alpha_1 > 0\) and \(\alpha_2 > 0\), \(c_m\) denotes the constant target contribution rate, \(x_p\) denotes the constant target funding level.
By the equation of optimality (5.1), we have that
\[-J_t = \min_{u_t,w} \left\{ e^{-\beta t} \left[ \alpha_1 (u_t - c_m)^2 + \alpha_2 (wx - x_p)^2 \right] + (bx + u_t - B) J_x + \left( \mu - b + \frac{3 - r_1 - r_2}{4} \right) wx J_x \right\} \]
\[= \min_{u_t,w} L(u_t, w) \]
where \( L(u_t, w) \) represents the term in the braces.

Denote \( k = \mu - b + (3 - r_1 - r_2) \sigma_2 / 4 \). The optimal \((u_t, w)\) satisfies
\[\frac{\partial L(u_t, w)}{\partial u_t} = 2\alpha_1 e^{-\beta t} (u_t - c_m) + J_x = 0, \]
and
\[\frac{\partial L(u_t, w)}{\partial w} = 2\alpha_2 e^{-\beta t} x (wx - x_p) + k x J_x = 0 \]
or
\[u_t = c_m - \frac{1}{2\alpha_1} J_x e^{\beta t}, \quad w = \frac{1}{x} \left( x_p - \frac{1}{2\alpha_2} k J_x e^{\beta t} \right). \]
Substitute them into (6.1) to get
\[-J_t = e^{-\beta t} \left[ \alpha_1 \left( -\frac{1}{2\alpha_1} J_x e^{\beta t} \right)^2 + \alpha_2 \left( -\frac{1}{2\alpha_2} k J_x e^{\beta t} \right)^2 \right] + \left( bx + c_m - \frac{1}{2\alpha_1} J_x e^{\beta t} - B \right) J_x + k \left( x_p - \frac{1}{2\alpha_2} k J_x e^{\beta t} \right) J_x \]
or
\[-J_t e^{\beta t} = (bx + c_m - B + k x_p) J_x e^{\beta t} - \frac{1}{4} \left( \frac{1}{\alpha_1} \right) \left( J_x e^{\beta t} \right)^2. \]

Denote \( a = c_m - B + k x_p, \ h = (1/\alpha_1 + k^2/\alpha_2) / 4 \). Then we have
\[-J_t e^{\beta t} = (bx + a) J_x e^{\beta t} - h (J_x e^{\beta t})^2 \]
and we conjecture that \( J(t, x) = e^{-\beta t} (px^2 + qx + r) \). Then
\[J_t = -\beta e^{-\beta t} (px^2 + qx + r), \quad J_x = e^{-\beta t} (2px + q). \]

Substituting them into (6.2) yields
\[\beta (px^2 + qx + r) = (bx + a)(2px + q) - h (2px + q)^2 \]
or
\[(4hp^2 + (\beta - 2b)p)x^2 + [4hpq - 2ap + (\beta - b)q]x + hq^2 - aq + \beta r = 0. \]

Then we find
\[p = \frac{2b - \beta}{4h}, \quad q = \frac{(2b - \beta)a}{2bh}, \quad r = \frac{a}{\beta} - \frac{a^2 (2b - \beta)^2}{4b^2 h}. \]

Therefore the optimal contribution rate and the asset allocation proportion is determined, respectively, by
\[u_t = c_m - \frac{2b - \beta}{4\alpha_1 h} \left( \frac{x}{2} + \frac{a}{b} \right), \quad w = \frac{1}{x} \left[ x_p - \frac{k(2b - \beta)}{4\alpha_2 h} \left( \frac{x}{2} + \frac{a}{b} \right) \right]. \]
7 Conclusion

Based on the concepts of canonical process and $V$ jump uncertain process, this paper provided an uncertain optimal control model with jump. The principle of optimality and the equation of optimality for uncertain optimal control with jump were obtained. As the applications of the equation of optimality, a pension funds control model was discussed and the optimal policies were analytically derived. The results show that the $V$ jump uncertain process is efficient for dealing with uncertain optimal control with jump.

References