Abstract: Game theory is a collection of mathematical models to study the behaviors of people with interest conflict, and has been applied extensively to economics, sociology, etc. However, in real-world situations, the players often lack the information about the other players’ (or even his own) payoffs, which leads to the games in uncertain environments. Within the framework of uncertainty theory, this paper investigates the uncertain-payoff two-player nonzero-sum game. Due to the uncertain payoffs, we introduce three decision criteria to define the behaviors of players, which leads to three types of games. For each type, we present a new definition of Nash equilibrium, its existence theorem, as well as one sufficient and necessary condition that provide a way to find such a Nash equilibrium. Finally, we give an example for illustrating the usefulness of the theory developed in this paper.

Keywords: Game theory; uncertain variable; expected value; uncertain measure; uncertain programming; Nash equilibrium

1 Introduction

Since von Neumann and Morgenstern’s seminal work [45], game theory has been used extensively to analyze conflict and cooperative situations in economics, sociology, etc. One of the fundamental problems in the game theory is the two-player nonzero-sum game. In such a game, each player strives to achieve as large a payoff as possible by choosing one from his pure strategy set. A mixed strategy describes a situation that each player, rather than choosing a particular pure strategy, will randomly select a pure strategy based on a given distribution. Nash [44] proved that there exists at least one (mixed) strategy, called Nash equilibrium, such that no player can improve his expected payoff by changing his strategy unilaterally.

In the literature, the nonzero-sum game with complete information has been discussed at length. However, as pointed out by Harsanyi [14], players in a real game often lack the information about the other players’ (or even his own) payoff functions. Furthermore, games are human-participated systems that contain paramenters expressed by subjective-intuitive opinion of people. For instance, two competing firms are marketing the same new product. The goal of each firm is to attract as many customers as possible by choosing one from its marketing strategy set that includes TV ad., free samples etc. In order to estimate the demand of the new product, we must rely on the subjective-intuitive opinions of experts with rich managerial experiences because there is no past statistical evidence and the effects of the firms’ marketing strategies on payoffs are very difficult to evaluate. For example, the demand quantity may be expressed by human language like “about 100 thousand units”. Then, how to deal with such subjective uncertainty is a basic problem for further discussion of uncertain game.

It is well-known that probability theory is a branch of mathematics concerned with analysis of randomness. In order to determine the probability distribution or density function of a random variable, it needs enough historical data for probabilistic reasoning. But things often go contrary to our wishes. Without enough data, people resort to the concept of fuzzy set that was initiated by Zadeh [51] to formulate the so called fuzziness. For the same purpose, uncertainty theory was founded [29] by Liu in 2007 and refined by Liu [34] in 2010. Uncertainty theory is a branch of mathematics based on an axiomatic system, like probability theory. By the uncertain statistics, the demand quantity can be formulated as an uncertain variable with an uncertainty distribution. Now, one may wonder, what on earth is the difference of the three theories, and which theory should we employ to model and analyze the uncertain game. Firstly, what distinguishes the probability, credibility and uncertainty theory is the measure of union of events. The probability measure is at one extreme, the sum of the measure of disjoint events. The credibility measure is at another extreme, the maximality of the measure of different events. While the uncertainty measure is an compromise of them satisfying the subadditivity property. It is obvious that an uncertain game is a human-participated system but not a physical system with additivity. Thus, we give up the probability theory also for lack of history data. Secondly, fuzzisit assume the demand for the new product to be fuzzy set, say, a triangular number (90, 100, 110). According to the fuzzy set theory, the event of demand’s being exactly 100 holds with possibility measure 1 (or credibility measure 0.5). On the other
hand, the opposite event of “not exactly 100” has the same possibility measure (or credibility measure). There is no doubt that nobody can accept this conclusion. This paradox shows that such imprecise quantities cannot be fuzzy concepts. Yet, we know that these quantities behave neither like randomness nor like fuzziness. Then uncertainty theory is the best choice to model these quantities for it overcomes the problems causes by both probability theory and possibility/credibility theory.

In literature, when payoffs were assumed to be random variables, and the resulting random-payoff games were solved by probabilistic methods (e.g., Blau [3], Cassidy et al [8], Charnes et al [9] and Harsanyi [13]). When the payoffs were assumed to be fuzzy sets, and the fuzzy-payoff games were developed by Aubin [1] and Butnariu [4] around 1980. Subsequently, many researchers have analyzed Nash equilibria in the framework of fuzzy set theory (Campos et al [5]-[6], Nishizaki and Sakawa [46]-[48], Maeda [41],[42]) and in the framework of credibility theory (Gao and his co-workers [15]-[21]).

In this paper, we investigate uncertain-payoff two-player nonzero-sum games (UTNGs) based on the uncertainty theory. In Section 2, we briefly call on some basic results of uncertainty theory, and introduce three decision criteria to be used for defining the behaviors of players. In Section 3, we describe the UTNGs and summarize four problems on the way to successfully solve the UTNGs. In Section 4-6, we study three types of UTNGs that are resulted from different decision criteria. For each, we define the best responses of a player as the set of optimal solutions of an uncertain programming problem. Then, we present a new definition of Nash equilibrium as well as its existence. We also present a sufficient and necessary condition, which shows that a Nash equilibrium combined with the outcome of the UTNG is an optimal solution to a quadratic programming problem with global maximum of zero. Finally, we provide an example for illustrating purpose in Section 7.

2 Preliminaries

Uncertainty theory was founded by Liu [29] in 2007 and refined by Liu [34] in 2010. Nowadays, uncertainty theory has become a branch of mathematics including uncertain process [30], uncertain calculus [33], uncertain differential equation [35], uncertain logic [24], uncertain inference [35], uncertain risk analysis [36]. Meanwhile, as an application of uncertainty theory, Liu [11] proposed a spectrum of uncertain programming with diverse applications. Other references related to uncertainty theory are Gao [19], Gao, et al [23], Liu and Xu [39], Peng and Iwamura [49], Liu and Ha [40]. For a detailed exposition of the uncertainty theory, the readers may consult the book [34].

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra on $\Gamma$. Then each element $\Lambda$ in $\mathcal{L}$ is called an event. Uncertain measure is a function from $\mathcal{L}$ to $[0, 1]$ subject to the following four axioms [29]:

**Normality Axiom:** $M\{\Gamma\} = 1$ for the universal set $\Gamma$.

**Monotonicity Axiom:** $M\{\Lambda_1\} \leq M\{\Lambda_2\}$ whenever $\Lambda_1 \subseteq \Lambda_2$.

**Self-Duality Axiom:** $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.

**Countable Subadditivity Axiom:** For every countable sequence of events $\{\Lambda_i\}$, we have

$$M \left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$  

**Product Measure Axiom:** Let $\Gamma_k$ be nonempty sets on which $M_k$ are uncertain measures, $k = 1, 2, \cdots, n$, respectively. Then the product uncertain measure $M$ is an uncertain measure on the product $\sigma$-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying

$$M\left\{ \left( \prod_{k=1}^{n} \Lambda_k \right) \right\} = \min_{1 \leq k \leq n} M_k\{\Lambda_k\},$$

where $\Lambda_k \in \mathcal{L}_k$, $k = 1, 2, \cdots, n$.

**Definition 2.1** (Liu [29]) An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

The concept of uncertain variable is distinguished from either random variable or fuzzy variable for it defined on a different measure space. The concept is a powerful tool for describing imprecise quantities in human systems.

**Definition 2.2** (Liu [29]) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = M\{\xi \leq x\}$$

for any real number $x$.

**Theorem 2.1** (Peng and Iwamura [49]) A function $\Phi : \mathbb{R} \rightarrow [0, 1]$ is an uncertainty distribution if and only if it is an increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. 

Theorem 2.2 (Liu [34]) Let $\xi$ be an uncertain variable with continuous uncertainty distribution $\Phi$. Then for any real number $x$, we have
\[ M\{\xi \leq x\} = \Phi(x), \quad M\{\xi \geq x\} = 1 - \Phi(x). \] (3)

Theorem 2.3 (Liu [34]) Let $\xi$ be an uncertain variable with continuous uncertainty distribution $\Phi$. Then for any interval $[a, b]$, we have
\[ \Phi(b) - \Phi(a) \leq M\{a \leq \xi \leq b\} \leq \Phi(b) \wedge (1 - \Phi(a)). \] (4)

Definition 2.3 (Liu [34]) An uncertainty distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. And $\Phi^{-1}$ is called the inverse uncertainty distribution of $\xi$.

It is easy to verify that a regular uncertainty distribution $\Phi$ is a continuous function. In addition, $\Phi$ is strictly increasing at each point $x$ with $0 < \Phi(x) < 1$. Furthermore,
\[ \lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1. \] (5)

In this paper, we will assume all uncertainty distributions are regular. Otherwise, we may give the uncertainty distribution a small perturbation such that it becomes regular.

Lemma 2.1 (Liu [34]) Let $\xi$ and $\eta$ be independent uncertain variables. Then for any real numbers $a$ and $b$, we have
\[ E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \] (6)

Definition 2.4 (Liu [34]) Let $\xi$ be an uncertain variable, and $\alpha \in (0, 1)$. Then
\[ \xi_{\text{sup}}(\alpha) = \sup\{r \mid M\{\xi \geq r\} \geq \alpha\} = \phi^{-1}(1 - \alpha) \] (7)
is called the $\alpha$-optimistic value to $\xi$.

Lemma 2.2 (Liu [34]) Let $\xi$ and $\eta$ be independent uncertain variables, and $\alpha \in (0, 1)$. Then for any nonnegative real numbers $a$ and $b$, we have
\[ (a\xi + b\eta)_{\text{sup}}(\alpha) = a\xi_{\text{sup}}(\alpha) + b\eta_{\text{sup}}(\alpha). \] (8)

**Definition 2.5** Let $\xi$, $\eta$ be two uncertain variables, $\alpha \in [0, 1]$, and $r$ be a real number. Then we have the following three ranking methods:
\[ \xi \geq \eta \iff E[\xi] \geq E[\eta], \]
\[ \xi \geq \eta \iff \xi_{\text{sup}}(\alpha) \geq \eta_{\text{sup}}(\alpha), \]
\[ \xi \geq \eta \iff M\{\xi \geq r\} \geq M\{\eta \geq r\}. \] (9)

### 3 Uncertain-Payoff Two-Player Nonzero-Sum Game

Let $M = \{1, 2, \cdots, m\}$ be the pure strategy set of one player $I$, and $N = \{1, 2, \cdots, n\}$ be the pure strategy set of the other player $J$. Formally, a two-person nonzero-sum game is any $\Gamma$ of the form
\[ \Gamma = \langle \{I, J\}, M \times N, A, B \rangle \]
where $A$ and $B$ are $m \times n$ matrices, whose entries $\xi_{ij}$ and $\eta_{ij}$ represent the payoffs of players $I$ and $J$ associated with the strategy profile $(i, j)$, respectively.

A mixed strategy of each player is in essence a probability distribution on his pure strategy set, which aims to describe a situation that the player, rather than choosing a particular pure strategy, will randomly select a pure strategy based on the given distribution. Denote the sets of all mixed strategies available for players $I$ and $J$ by
\[ S_I = \left\{ (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m_+ \mid \sum_{i=1}^m x_i = 1 \right\} \]
\[ S_J = \left\{ (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n_+ \mid \sum_{j=1}^n y_j = 1 \right\}. \]

Then a mixed strategy game is any $\Gamma'$ of the form
\[ \Gamma' = \langle \{I, J\}, S_I \times S_J, A, B \rangle. \]

When the two-person nonzero-sum game $\Gamma'$ is played, players $I$ and $J$ must choose one mixed strategy from their own strategy set, say $x$ and $y$, respectively. Then, the strategy profile $(x, y)$ determines the outcome of the game $(x^T Ay, x^T By)$, where $x^T Ay$ and $x^T By$ are the expected payoffs of player $I$ and $J$, respectively. A two-person nonzero-sum game is finite if each player has a finite set of strategies. For finite two-person nonzero-sum games, a well-known result is given as follows.

**Theorem 3.1** (Nash [44]) There exists at least one mixed strategy Nash equilibrium in a finite two-person nonzero-sum game.
In a real game, the decision environment is often characterized by a large number of possible strategies, complicated relations between strategic choices and their intricate influences to payoffs, thus making accurate or probabilistic estimation of the payoff matrices impossible. In such a situation, we may specify the payoffs as uncertain variables via uncertain statistics. Now we denote the uncertain payoff matrices of players \( I \) and \( J \) by \( \tilde{A} \) and \( \tilde{B} \), respectively. For any mixed strategy profile, say \((x, y)\), the expected payoffs \( x^T \tilde{A} y \) and \( x^T \tilde{B} y \) are uncertain, too. In order to analyze the UTNGs, we should consider the following problems:

1. What is the optimal strategy of a player given the other one’s?
2. How to define the Nash equilibrium?
3. Does the Nash equilibrium exist?
4. How to find a Nash equilibrium?

Many researchers have tried to cope with the above problems based on probability theory and fuzzy set theory. The attempt of this paper is to establish a theoretical framework for the analysis of UTNGs based on the uncertainty theory. In the following sections, three types of UTNGs are to be investigated, and the above four questions are to be tackled.

4 The Expected Nash Equilibrium Strategy

Liu [29][34] proposed the expected value operator of uncertain variable and uncertain expected value model. Assuming that the players’ goals are to maximize the expected value of their uncertain objectives in the decision process, we have the first type of UTNGs of the form

\[
\tilde{\Gamma}_E = \langle \{I_E, J_E\}, S_I \times S_J, \tilde{A}, \tilde{B} \rangle,
\]

where \( \{I_E, J_E\} \) represents that the players both adopt the expected value criterion.

In the UTNG \( \tilde{\Gamma}_E \), the best responses of player \( I \) to a strategy \( y^* \in S_J \) are the optimal solutions of the uncertain expected value model

\[
\max_{x \in S_I} E \left[ x^T \tilde{A} y^* \right],
\]

(10)

and the best responses of player \( J \) to a strategy \( x^* \in S_I \) are the optimal solutions of the uncertain expected value model

\[
\max_{y \in S_J} E \left[ x^T \tilde{B} y \right].
\]

(11)

Then, based on the rational reactions of the players in \( \tilde{\Gamma}_E \), we present a new definition of Nash equilibrium strategy as follows.

**Definition 4.1** An array \((x^*, y^*)\) is called an expected Nash equilibrium strategy (ENES), if it satisfies

\[
u^* = E \left[ x^T \tilde{B} y^* \right] \geq E \left[ x^T \tilde{B} y \right] \forall y \in S_J.
\]

The pair \((u^*, v^*)\) is called the expected value of the game.

Given the concept of ENES, one may be concerned with the existence of an ENES in \( \tilde{\Gamma}_E \), which is shown in the following theorem.

**Theorem 4.1** Let all entries \( \xi_{ij} \) of the payoff matrix \( \tilde{A} \) be independent uncertain variables, and all entries \( \eta_{ij} \) of the payoff matrix \( \tilde{B} \) be independent uncertain variables. Then there exists at least one ENES in the \( \tilde{\Gamma}_E \). Let \((x^*, y^*) \in S_I \times S_J \) be an ENES. Then the expected value of the game is

\[
(x^T \Delta y^*, x^T \nabla y^*),
\]

where

\[
\Delta = \begin{bmatrix}
E[\xi_{11}] & E[\xi_{12}] & \cdots & E[\xi_{1n}]
E[\xi_{21}] & E[\xi_{22}] & \cdots & E[\xi_{2n}]
\vdots & \vdots & \ddots & \vdots
E[\xi_{m1}] & E[\xi_{m2}] & \cdots & E[\xi_{mn}]
\end{bmatrix},
\]

(12)

\[
\nabla = \begin{bmatrix}
E[\eta_{11}] & E[\eta_{12}] & \cdots & E[\eta_{1n}]
E[\eta_{21}] & E[\eta_{22}] & \cdots & E[\eta_{2n}]
\vdots & \vdots & \ddots & \vdots
E[\eta_{m1}] & E[\eta_{m2}] & \cdots & E[\eta_{mn}]
\end{bmatrix},
\]

(13)

Now, we give a sufficient and necessary condition of an ENES, which provides a way to find one ENES by solving a quadratic programming problem with the global maximum of zero.

**Theorem 4.2** Let all entries \( \xi_{ij} \) of the payoff matrix \( \tilde{A} \) be independent uncertain variables, and all entries \( \eta_{ij} \) of the payoff matrix \( \tilde{B} \) be independent uncertain variables. Then a strategy \((x^*, y^*) \in S_I \times S_J \) is an ENES in \( \tilde{\Gamma}_E \) if and only if the point \((x^*, y^*, x^T \Delta y^*, x^T \nabla y^*)\) is an optimal solution to the following quadratic programming problem,

\[
\begin{align*}
\max_{x,y,u,v} & \quad x^T (\Delta + \nabla) y - u - v \\
\text{subject to:} & \quad \Delta y \leq (u, u, \ldots, u)^T \\
& \quad \nabla^T x \leq (v, v, \ldots, v)^T \\
& \quad \forall x \in S_I, y \in S_J, u, v \in \mathbb{R}
\end{align*}
\]

(14)

where \( \Delta \) and \( \nabla \) are defined in equations (12) and (13).
5 The Optimistic Nash Equilibrium Strategy

In the area of decision-making under uncertainty, the second popular decision criterion is to optimize an α-optimistic value of the uncertain objective function, where α is a predetermined confidence level. This decision criterion is mostly used in the chance-constrained programming, which was proposed by Charnes and Cooper [10] and extended to uncertain optimization problem by Liu [34]. Assuming that the players’ goals are to maximize the optimistic values of their uncertain objectives at given confidence level α and β, respectively, we have the second type of UTNGs of the form

\[ \tilde{\Gamma}_O(\alpha, \beta) = \langle \{I_\alpha, J_\beta\}, S_I \times S_J, \tilde{A}, \tilde{B} \rangle, \]

where \(\{I_\alpha, J_\beta\}\) represents that player I adopts the optimistic value criterion with confidence level α, and player J adopts the optimistic value criterion with confidence level β. In the following, we analyze \(\tilde{\Gamma}_O(\alpha, \beta)\) by solving the problems proposed in Section 3 in turn.

In the game \(\tilde{\Gamma}_O(\alpha, \beta)\), the best responses of player I to a strategy \(y^* \in S_J\) are the optimal solutions of the uncertain chance-constrained programming model

\[
\max_{x \in S_I} \max_{y \in S_J} M \left\{ x^T \tilde{A} y^* \geq u \right\} \geq \alpha, \tag{15}
\]

and the best responses of player J to a strategy \(x^* \in S_I\) are the optimal solutions of the uncertain chance-constrained programming model

\[
\max_{y \in S_J} \max_{v \in S_J} M \left\{ x^*^T \tilde{B} y \geq v \right\} \geq \beta. \tag{16}
\]

Then, based on the rational reactions of the players, we present another new definition of Nash equilibrium strategy of \(\tilde{\Gamma}_O(\alpha, \beta)\).

**Definition 5.1** An array \((x^*, y^*)\) is called an optimistic Nash equilibrium strategy (ONES) of \(\tilde{\Gamma}_O(\alpha, \beta)\), if it satisfies

\[
u^* = \max \left\{ u \mid M \left\{ x^*^T \tilde{A} y^* \geq u \right\} \geq \alpha \right\}, \quad \forall x \in S_I,
\]

\[
v^* = \max \left\{ v \mid M \left\{ x^*^T \tilde{B} y \geq v \right\} \geq \beta \right\}, \quad \forall y \in S_J.
\]

The point \((u^*, v^*)\) is called the optimistic value of the game.

Once the ONES is defined, one may be concerned with the existence of an ONES in \(\tilde{\Gamma}_O(\alpha, \beta)\). The following theorem will give an answer.

**Theorem 5.1** Let \(\xi_{ij}, \eta_{ij}\) in the payoff matrices \(\tilde{A}\) and \(\tilde{B}\) be independent uncertain variables. Then there exist at least one ONES in the game \(\tilde{\Gamma}_O(\alpha, \beta)\). Let \((x^*, y^*) \in S_I \times S_J\) be one ONES of \(\tilde{\Gamma}_O(\alpha, \beta)\). Then the optimistic value of the game is

\[
(x^*^T \tilde{A}_{sup} y^* + x^*^T \tilde{B}_{sup} y^*)
\]

where

\[
\tilde{A}_{sup} = \begin{bmatrix}
\xi_{11}^{\alpha} & \xi_{12}^{\alpha} & \cdots & \xi_{1n}^{\alpha} \\
\xi_{21}^{\alpha} & \xi_{22}^{\alpha} & \cdots & \xi_{2n}^{\alpha} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{m1}^{\alpha} & \xi_{m2}^{\alpha} & \cdots & \xi_{mn}^{\alpha}
\end{bmatrix}, \tag{17}
\]

\[
\tilde{B}_{sup} = \begin{bmatrix}
\eta_{11}^{\beta} & \eta_{12}^{\beta} & \cdots & \eta_{1n}^{\beta} \\
\eta_{21}^{\beta} & \eta_{22}^{\beta} & \cdots & \eta_{2n}^{\beta} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{m1}^{\beta} & \eta_{m2}^{\beta} & \cdots & \eta_{mn}^{\beta}
\end{bmatrix}. \tag{18}
\]

**Theorem 5.2** Let \(\xi_{ij}, \eta_{ij}\) in the payoff matrices \(\tilde{A}\) and \(\tilde{B}\) be independent uncertain variables. Then a strategy \((x^*, y^*) \in S_I \times S_J\) is an ONES of \(\tilde{\Gamma}_O(\alpha, \beta)\) if and only if the point

\[
(x^*, y^*, x^*^T \tilde{A}_{sup} y^* + x^*^T \tilde{B}_{sup} y^*)
\]

is an optimal solution to the following quadratic programming problem,

\[
\begin{align*}
\max_{x, y, u, v} & \quad x^T (\tilde{A}_{sup} + \tilde{B}_{sup}) y - u - v \\
\text{subject to:} & \quad \tilde{A}_{sup} y \leq (u, u, \ldots, u)^T \\
& \quad \tilde{B}_{sup} x \leq (v, v, \ldots, v)^T \\
& \quad \forall x \in S_I, \ y \in S_J, \ u, v \in \mathbb{R}
\end{align*} \tag{19}
\]

where \(\tilde{A}_{sup}\) and \(\tilde{B}_{sup}\) are defined in Equation (17) and (18), respectively.

6 The Maximal Chance Nash Equilibrium Strategy

In many cases, a decision maker may be concerned with an event (e.g., objective function’s being greater than a prospective value) and wish to maximize the chance of the event. This decision criterion was introduced and used in uncertain dependent-chance programming [34]. Assuming that each player has specified a prospective value, and wants to maximize the credibility of his uncertain total payoff’s achieving the prospective value, we have the third type of UTNGs of the form

\[
\tilde{\Gamma}_M(u, v) = \langle \{I_u, J_v\}, S_I \times S_J, \tilde{A}, \tilde{B} \rangle,
\]
where \( \{I_u, J_v\} \) represents that the players set their profit level as \( u \) and \( v \), respectively. In the following, we analyze \( \Gamma_M(u, v) \) by solving the problems proposed in Section 3 one by one.

In the game \( \Gamma_M(u, v) \), the best responses of the player \( I \) to a strategy \( y^* \in S_J \) are the optimal solutions of the following uncertain dependent-chance programming model:

\[
\max_{x \in S_I} M \left\{ x^T \tilde{A} y^* \geq u \right\}.
\]

Similarly, after specified a prospective value \( v \), player \( J \) tries to maximize the credibility of his uncertain total payoff’s being greater than \( v \), then the best responses of the player \( J \) to a strategy \( x^* \in S_I \) are the optimal solutions of the following uncertain dependent-chance programming model:

\[
\max_{y \in S_J} M \left\{ x^T \tilde{B} y \geq v \right\}.
\]

Then, based on the rational actions of the players, we present the definition of Nash equilibrium strategy of \( \tilde{\Gamma}_M(u, v) \).

**Definition 6.1** An array \((x^*, y^*)\) is called a maximal chance Nash equilibrium strategy (MNES) of \( \tilde{\Gamma}_M(u, v) \), if it satisfies

\[
\alpha = M \left\{ x^T \tilde{A} y^* \geq u \right\} \geq M \left\{ x^T \tilde{A} y^* \geq u \right\} \quad \forall x \in S_I,
\]

\[
\beta = M \left\{ x^T \tilde{B} y \geq v \right\} \geq M \left\{ x^T \tilde{B} y \geq v \right\} \quad \forall x \in S_J.
\]

The point \((\alpha, \beta)\) is called the maximal chance of the game \( \tilde{\Gamma}_M(u, v) \).

**Theorem 6.1** Let \( \xi_{ij}, \eta_{ij} \) be the payoff matrices \( \tilde{A} \) and \( \tilde{B} \) be independent uncertain variables. Then, for any \( u \in \mathbb{R} \) and \( v \in \mathbb{R} \), there exist at least one MNES in the game \( \tilde{\Gamma}_M(u, v) \). Let \((x^*, y^*) \in S_I \times S_J \) be an MNES. Then the maximal chance of the game is \((\alpha, \beta)\), where

\[
\alpha = M \left\{ x^T \tilde{B} y^* \geq u \right\}, \beta = M \left\{ x^T \tilde{B} y^* \geq v \right\}.
\]

Now we present a sufficient and necessary condition for an MNES.

**Theorem 6.2** Let \( \xi_{ij}, \eta_{ij} \) be the payoff matrices \( \tilde{A} \) and \( \tilde{B} \) be independent uncertain variables. Then a strategy \((x^*, y^*) \in S_I \times S_J \) is an MNES of \( \tilde{\Gamma}_M(u, v) \) if and only if the point

\[
(x^*, y^*, M \left\{ x^T \tilde{A} y^* \geq u \right\}, M \left\{ x^T \tilde{B} y^* \geq v \right\})
\]

is an optimal solution to the following quadratic programming problem,

\[
\begin{align*}
\max_{x, y, \alpha, \beta} & \quad M \left\{ x^T \tilde{B} y \geq u \right\} - \alpha \\
& \quad + M \left\{ x^T \tilde{B} y \geq v \right\} - \beta \\
\text{subject to:} & \quad M \left\{ \tilde{A} y \geq (u, u, \cdots, u)^T \right\} \leq \alpha \tag{22} \\
& \quad M \left\{ \tilde{B}^T x \geq (v, v, \cdots, v)^T \right\} \leq \beta \\
& \quad x \in S_I, \quad y \in S_J, \quad \alpha, \beta \in [0, 1].
\end{align*}
\]

7 An Example

Consider a situation in which two competing firms are marketing the same new product. The goal of each firm is to attract as many customers as possible by choosing one from its marketing strategy set including TV ad., free samples, special offers, cash rebate etc. A mixed strategy determines how to divide its budget among its various marketing alternatives.

Very often there are other products with similar functions in the market. Different marketing strategies of the two firms may lead to different total demand of the product. We assume that the demand for the product is variable, and the two firms are in the situation of a nonzero-sum game. For simplicity, we assume that each firm has only two marketing alternatives. That is, the pure strategy set of player \( I \) and \( J \) are \( M = \{1, 2\} \) and \( N = \{1, 2\} \), respectively. For the lack of past statistical evidence about the demand for the new product, the two firms have to take use of their experiences, subjective judgement and intuitions of managers, and determine that \( \xi_{ij} \) and \( \eta_{ij} \) are characterized by triangular uncertain numbers as follows:

\[
\begin{align*}
\xi_{11} &= (90, 100, 110), \quad \xi_{12} = (110, 160, 170) \\
\xi_{21} &= (130, 140, 190), \quad \xi_{22} = (60, 110, 120) \\
\eta_{11} &= (80, 105, 110), \quad \eta_{12} = (110, 150, 190) \tag{22} \\
\eta_{21} &= (120, 140, 200), \quad \eta_{22} = (80, 90, 140),
\end{align*}
\]

where the \( \xi_{ij} \) and \( \eta_{ij} \) are expressed in units of thousands.

Firstly, we assume that the firms both adopt the expected value criterion, which gives

\[
\Delta = \nabla = \left[ \begin{array}{cc} 100 & 150 \\ 150 & 100 \end{array} \right].
\]

Based on Theorem 4.2, we solve the quadratic programming (14) and obtain an optimal solution \((0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)\). That is, the vector \((0.5, 0.5, 0.5, 0.5)\) is an ENES, and the optimal strategy of either player is \((0.5, 0.5)\) yielding an expected payoff 125.

Secondly, we assume that the firms both want to attract as many customer as possible with a confidence level 0.8. This leads to

\[
\begin{align*}
\tilde{A}^\circ_{\text{sup}} &= \left[ \begin{array}{cc} 94 & 130 \\ 134 & 80 \end{array} \right], \quad \tilde{B}^\circ_{\text{sup}} = \left[ \begin{array}{cc} 90 & 126 \\ 128 & 84 \end{array} \right].
\end{align*}
\]
After solving a quadratic programming problem defined in (19), we get an optimal solution 
\((0.55, 0.45, 0.56, 0.44, 110, 107)\) with optimum zero. That is, the optimal strategy for firm \(I\) is \(x^* = (0.55, 0.45)\) yielding a payoff level 110,000 with credibility 80%; and the optimal strategy for firm \(J\) is \(x^* = (0.56, 0.44)\) yielding a payoff level 107,000 with credibility 80%; and 
\((0.55, 0.45, 0.56, 0.44)\) is an ONES of \(\tilde{\Gamma}_O(0.8, 0.8)\). Obviously, we can select different parameter settings, for example \((0.6, 0.7), (0.9, 0.8)\) etc. And the problem can be solved by a similar procedure.

Thirdly, we can also deduce that the vector 
\((0.55, 0.45, 0.56, 0.44)\) is also an MNES of \(\tilde{\Gamma}_M(110, 107)\). That is, if the forms select 110 and 107 as their target values respectively, then the optimal strategy of firm \(I\) is \((0.55, 0.45)\) with his chance of achieving the target value 0.8; and the optimal strategy of firm \(J\) is \((0.56, 0.44)\) with his chance of achieving the target value 0.8. When the firms select other parameter settings, for example \((120, 125), (130, 120)\) etc, we can solve the quadratic programming problem defined by (22), and find an MNES as well as the most credibility of the game associated with it.

8 Conclusion

In this paper, we presented a theoretical framework, which enables us to analyze UTNGs like auction, web-e-commerce etc. We emphasize that with this framework, we can formulate UTNGs as mathematical models and give mathematical definitions of Nash equilibria according to different decision criteria in uncertain environments. With different parameter settings (e.g., the confidence levels, target values of players), we can describe the decision situations of players with more preciseness. Moreover, the existence theorems ensure the significance of our new models and definitions of Nash equilibria, and the sufficient and necessary conditions provide ways to find the Nash equilibria.

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