Continuous dependence theorems on solutions of uncertain differential equations

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Abstract

In ordinary differential equation (ODE) and stochastic differential equation (SDE), the solution continuously depends on initial value and parameter under some conditions. This paper investigates the analogous continuous dependence theorems in uncertain differential equation (UDE). It proves two continuous dependence theorems, a basic one and a general one.

1. Introduction

Nondeterministic phenomenon in dynamic system, such as perturbation and white noise, is usually described by random variable. As a result, stochastic differential equation becomes the main means to research dynamic system with perturbation. However, lots of surveys show that sometimes it is not suitable to regard the perturbation as random variable. For example, the data describing the dynamic system may come from expert, especially in researching economic or social system, and we can not regard the expert data as random variable. This means that SDE can not model all dynamic systems.

How can we deal with dynamic systems with expert data? In order to solve this problem, Liu proposed uncertainty theory [1,2] and uncertain differential equation [3]. Up to now, a lot of research on UDE has been started. Chen and Liu [4] and Gao [5] proved a series existence and uniqueness theorems on UDE. Yao and Chen [6] gave a numerical method to solve UDE. Zhu [7] introduced UDE into optimal control. Some researchers employed UDE to model financial market, such as Peng and Yao [8] and Chen [9].

This paper studies how the solution of UDE depends on initial value and parameter in coefficient. In ODE [10,11] and SDE [12–14], the solution is continuous with respected to initial value and parameter under some conditions. Will the analogous conclusion hold in UDE? This paper gives an affirmative answer. It first proves a basic continuous theorem, which says if the coefficients of UDE satisfy global Lipschitz condition and linear growth condition, the solution is continuously depending on initial value and parameter.

Although the basic continuous theorem is easy to understand, the global Lipschitz condition is so strict that most UDEs can not satisfy it. To solve this problem, this paper further proves a general theorem. The strong theorem only requires that the coefficients are continuous and the solution is unique.
The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertainty theory and UDE used throughout this paper are introduced. In Section 3, the basic continuous theorem is proved. Section 4 proves the general continuous theorem. Section 5 gives a brief summary to this paper.

2. Preliminary concepts and definitions

In this section, we introduce some foundational concepts and properties of uncertainty theory and UDE, which are used throughout this paper.

Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in \mathcal{L} \) is assigned a number \( \mathcal{M}(\Lambda) \in [0, 1] \). In order to ensure that the number \( \mathcal{M}(\Lambda) \) has certain mathematical properties, Liu ([1, 2]) presented the four following axioms: (1) normality, (2) self-duality, (3) countable subadditivity, and (4) product measure axioms. If the first three axioms are satisfied, the set function \( \mathcal{M}(\Lambda) \) is called an uncertain measure.

**Definition 1** (Liu [1]). Let \( \Gamma \) be a nonempty set, \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \), and \( \mathcal{M} \) an uncertain measure. Then the triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space.

**Definition 2** (Liu [1]). An uncertain variable is a measurable function \( \xi \) from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set \[
\{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \},
\]
is an event.

**Definition 3** (Liu [3]). Let \( T \) be an index set and let \((\Gamma, \mathcal{L}, \mathcal{M})\) be an uncertainty space. An uncertain process is a measurable function from \( T \times (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers, i.e., for each \( t \in T \) and any Borel set \( B \) of real numbers, the set \[
\{ \xi_t \in B \} = \{ \gamma \in \Gamma | \xi_t(\gamma) \in B \},
\]
is an event.

That is, an uncertain process \( \xi_t(\gamma) \) is a function of two variables such that the function \( \xi_t(\gamma) \) is an uncertain variable for each \( t \).

**Definition 4** (Liu [15]). An uncertain process \( C_t \) is said to be a canonical process if

(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,
(ii) \( C_t \) has stationary and independent increments,
(iii) every increment \( C_{i+t} - C_t \) is a normal uncertain variable with expected value 0 and variance \( t^2 \).

An uncertain variable \( \xi \) is called normal if it has a normal uncertainty distribution
\[
\Phi(x) = \mathcal{M}(\xi \leq x) = \left( 1 + \exp \left( \frac{\pi (e - x)}{\sqrt{3 \sigma}} \right) \right)^{-1}, \quad x \in \mathbb{R},
\]
whose expected value is \( e \) and variance is \( \sigma^2 \).

**Definition 5** (Liu [3]). Suppose \( C_t \) is a canonical process, and \( f \) and \( g \) are some given functions. Then
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t,
\]
is called an uncertain differential equation. A solution is an uncertain process \( X_t \) that satisfies (1) identically in \( t \).

For the sake of simplicity, our discussion is not based on the form of UDE (1) but on its equivalent integral form, or say, uncertain integral equation
\[
X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s.
\]

In this paper, we investigate the general form of uncertain integral Eq. (2)
\[
X_t^{(p)} = X_0^{(p)} + \int_0^t f(s, X_s^{(p)}, p)ds + \int_0^t g(s, X_s^{(p)}, p)dC_s,
\]
that is, the initial value \( X_0^{(p)} \) and coefficients \( f(t, x, p) \) and \( g(t, x, p) \) all depend on parameter \( p \). Our goal is to find how the solution \( X_t^{(p)} \) depends on parameter \( p \) on a finite interval \([0, T]\).
3. A basic continuity theorem

In 2010, Chen and Liu [4] gave the first existence and uniqueness theorem, where they said if the coefficients of UDE (1) satisfy global Lipchitz condition and linear growth condition, there exists a unique solution. The conditions are strict, but the proof is easy. Here, we follow their method and discuss how the solution \( X_t^p \) depends on parameter \( p \), when \( f(t,x,p) \) and \( g(t,x,p) \) of uncertain integral Eq. (3) satisfy global Lipchitz condition and linear growth condition.

**Condition 1.** The coefficients \( f(t,x,p) \) and \( g(t,x,p) \) satisfy global Lipchitz condition and linear growth condition in \( x \), i.e., there exists a positive constant \( L \) such that

\[
|f(t,x_1,p) - f(t,x_2,p)| \leq L |x_1 - x_2|, \quad |g(t,x_1,p) - g(t,x_2,p)| \leq L |x_1 - x_2|,
\]

whenever \( t \in [0,T] \) and \( p \in [-1,1] \).

It is clear that under Condition 1, uncertain integral equation

\[
X_t^p = X_0^p + \int_0^t f(s,X_s^p,p)ds + \int_0^t g(s,X_s^p,p)dc_s.
\]

has a unique solution \( X_t^p \) on \( 0 \leq t \leq T \). We will prove that \( X_t^p \) uniformly converges to \( X_t^{(0)} \) on \( 0 \leq t \leq T \) almost surely.

**Theorem 1.** Under Condition 1, we have

\[
\lim_{p \to 0} \sup_{0 \leq t \leq T} |X_t^p - X_t^{(0)}| = 0, \quad \text{a.s.}
\]

**Proof.** According to the definition of canonical process \( C_t \), there exists a set \( \Gamma_0 \) with \( \mathcal{M}(\Gamma_0) = 0 \) such that for each \( \gamma \in \Gamma \setminus \Gamma_0 \), the path \( C_t(\gamma) \) is Lipchitz continuous with Lipchitz constant \( K(\gamma) \). Obviously, Theorem 1 holds if we can prove that for each \( \gamma \in \Gamma \setminus \Gamma_0 \),

\[
\lim_{p \to 0} \sup_{0 \leq t \leq T} |X_t^p(\gamma) - X_t^{(0)}(\gamma)| = 0.
\]

where \( X_t^p(\gamma) \) is the solution of uncertain integral equation

\[
X_t^p(\gamma) = X_0^p + \int_0^t f(s,X_s^p(\gamma),p)ds + \int_0^t g(s,X_s^p(\gamma),p)dc_s(\gamma).
\]

It is easy to verify that \( X_t^p(\gamma) \) is bounded on \( [0,T] \). Indeed, for any \( t \in [0,T] \),

\[
|X_t^p(\gamma)| \leq |X_0^p(\gamma)| + \int_0^t |f(s,X_s^p(\gamma),p)|ds + \int_0^t |g(s,X_s^p(\gamma),p)|dc_s(\gamma)
\]

\[
\leq |X_0^p(\gamma)| + \int_0^t |f(s,X_s^p(\gamma),p)|ds + K(\gamma) \cdot \int_0^t |g(s,X_s^p(\gamma),p)|ds
\]

\[
\leq |X_0^p(\gamma)| + \int_0^t |f(s,X_s^p(\gamma),p)|ds + K(\gamma) \cdot \int_0^t |g(s,X_s^p(\gamma),p)|ds
\]

\[
\leq (|X_0^p(\gamma)| + T(1 + K(\gamma))L) + (1 + K(\gamma))L \int_0^t |X_s^p(\gamma)|ds, \quad \forall t \in [0,T].
\]

Setting \( A_p = |X_0^p(\gamma)| + T(1 + K(\gamma))L \), by Gronwall inequality, we have

\[
|X_t^p(\gamma)| \leq A_p \cdot \exp\{(1 + K(\gamma))L\} \leq A_p \cdot \exp\{(1 + K(\gamma))L\} < +\infty, \quad \forall t \in [0,T],
\]

that is, \( X_t^p(\gamma) \) is bounded on \( [0,T] \).

We write

\[
X_t^p(\gamma) - X_t^{(0)}(\gamma) = Y_t^p(\gamma) + \int_0^t \left( f(s,X_s^p(\gamma),p) - f(s,X_s^{(0)}(\gamma),p) \right)ds + \int_0^t \left( g(s,X_s^p(\gamma),p) - g(s,X_s^{(0)}(\gamma),p) \right)dc_s(\gamma),
\]

where

\[
Y_t^p(\gamma) = X_t^p(\gamma) - X_t^{(0)}(\gamma) + \int_0^t \left( f(s,X_s^0(\gamma),p) - f(s,X_s^{(0)}(\gamma),p) \right)ds + \int_0^t \left( g(s,X_s^0(\gamma),p) - g(s,X_s^{(0)}(\gamma),p) \right)dc_s(\gamma).
\]

There exists a function \( H(p, \gamma) \) such that \( |Y_t^p(\gamma)| \leq H(p, \gamma) \), for any \( t \in [0,T] \), i.e.,
\[ |Y_t^{(p)}(\gamma)| \leq |X_0^{(p)}(\gamma) - X_0^{(0)}(\gamma)| + \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds + \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| dc_s(\gamma) \]

\[ \leq |X_0^{(p)}(\gamma) - X_0^{(0)}(\gamma)| + \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds + \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| dc_s(\gamma) \]

\[ = H(p, \gamma). \]

Then, we have

\[ |X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma)| \leq H(p, \gamma) + \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds + \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| dc_s(\gamma) \]

\[ \leq H(p, \gamma) + L \int_0^T \left| X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma) \right| ds + L \int_0^T \left| X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma) \right| dc_s(\gamma) \]

\[ \leq H(p, \gamma) + L \int_0^T \left| X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma) \right| ds + K(\gamma)L \int_0^T \left| X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma) \right| ds. \] (6)

by Gronwall inequality [16], we have

\[ |X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma)| \leq H(p, \gamma) \cdot \exp((1 + K(\gamma))L) \leq H(p, \gamma) \cdot \exp((1 + K(\gamma))L), \forall t \in [0, T]. \] (7)

We will prove that \( H(p, \gamma) \to 0 \) if \( p \to 0 \). Since \( X_t^{(0)}(\gamma) \) is bounded on \([0, T]\), \( f \) and \( g \) are continuous, by Lebesgue convergence theorem, we have

\[ \lim_{p \to 0} H(p, \gamma) = \lim_{p \to 0} |X_0^{(p)}(\gamma) - X_0^{(0)}(\gamma)| + \lim_{p \to 0} \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds \]

\[ + \lim_{p \to 0} \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| dc_s(\gamma) \]

\[ \leq \lim_{p \to 0} \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds + \lim_{p \to 0} \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| ds \]

\[ = \int_0^T \left| f(s, X_s^{(0)}(\gamma), p) - f(s, X_s^{(0)}(\gamma), 0) \right| ds + K(\gamma) \int_0^T \left| g(s, X_s^{(0)}(\gamma), p) - g(s, X_s^{(0)}(\gamma), 0) \right| ds = 0. \]

According to expression (7), we have

\[ \limsup_{p \to 0, t \in [0, T]} |X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma)| = 0. \]

The proof is completed. \( \square \)

Theorem 1 is a weak conclusion, since the global Lipschitz condition is too strict and few UDEs satisfy Condition 1. In most situations, we only concern about whether \( X_t^{(0)}(\gamma) \) is continuous at \( p = 0 \). However, Theorem 1 actually concerns about whether \( X_t^{(p)}(\gamma) \) is continuous at \( p \in [-1, 1]\). As a result, we may find a general continuity theorem, which is the main conclusion in the next section.

4. A general continuity theorem

Before giving the general continuity theorem, we introduce the Ascoli–Arzelà Theorem.

**Definition 6.** *(Equicontinuity).* On a bounded interval \([a, b]\), a sequence of functions \( \{f_n\}_{n=1}^\infty \) is said to be equicontinuous if for every \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) > 0 \) such that

\[ |f_n(t_1) - f_n(t_2)| < \varepsilon, \forall n \geq 1, |t_1 - t_2| < \delta(\varepsilon). \]

**Definition 7.** *(Uniform Boundedness).* On a bounded interval \([a, b]\), a sequence of functions \( \{f_n\}_{n=1}^\infty \) is said to be bounded if there exists a non-negative number \( B \) such that

\[ |f_n(t)| < B, \forall n \geq 1, \forall t \in [a, b]. \]

**Theorem 2.** *(Ascoli–Arzelà Theorem).* On a bounded interval \([a, b]\), every bounded and equicontinuous sequence of functions \( \{f_n\}_{n=1}^\infty \) contains a subsequence which is uniformly convergent on \([a, b]\).

Now, we give the condition of the general continuity theorem.

**Condition 2.** The uncertain integral equation
\[ X_t^{(p)} = X_0^{(p)} + \int_0^t f(s, X_s^{(p)}, p)ds + \int_0^t g(s, X_s^{(p)}, p)dC_s. \]

satisfies

(i) \( f(t, x, p) \) and \( g(t, x, p) \) are continuous in \((t, x, p)\) on \([0, T] \times \mathbb{R} \times [-1, 1] \),
(ii) \( X_t^{(p)} \) is the unique continuous solution of uncertain integral equation on \([0, T] \)

\[ X_t^{(0)} = X_0^{(0)} + \int_0^t f(s, X_s^{(0)}, 0)ds + \int_0^t g(s, X_s^{(0)}, 0)dC_s. \]

Obviously, Condition 1 is a special case of Condition 2.

**Theorem 3.** Under Condition 2, we have
\[
\lim_{p \to 0 \atop t \in [0, T]} |X_t^{(p)} - X_t^{(0)}| = 0, \text{ a.s.} \tag{4}
\]

**Proof.** Condition 2 indicates that there exists a positive number \( H \) such that
\[
|f(t, x, p)| \leq H, \quad |g(t, x, p)| \leq H, \quad \forall (t, x, p) \in \Delta \times [-1, 1],
\]
where \( \Delta = \{(t, x) : 0 \leq t \leq T, |x - X_t^{(0)}| \leq 1\} \) is a closed set. We will prove Theorem 3 by employing the following two lemmas.

**Lemma 1.** For each \( \gamma \in \Gamma \setminus \Gamma_0, \) if there exists a positive number \( p_j \leq 1 \) such that \((t, X_t^{(p_j)}) \in \Delta \text{ on } [0, T] \times (-p_j, p_j),\) we have
\[
\lim_{p \to 0 \atop t \in [0, T]} |X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma)| = 0. \tag{5}
\]

**Proof.** For each \( \gamma \in \Gamma \setminus \Gamma_0, \) when \( p \in (-p_j, p_j), \) we have
\[
|X_t^{(p)}(\gamma)| \leq |X_t^{(0)}(\gamma)| + \int_0^t |f(s, X_s^{(p)}(\gamma), p)|ds + \int_0^t |g(s, X_s^{(p)}(\gamma), p)|dC_s(\gamma)
\]
\[
\leq |X_t^{(0)}(\gamma)| + \int_0^t |f(s, X_s^{(p)}(\gamma), p)|ds + K(\gamma) \cdot \int_0^t |g(s, X_s^{(p)}(\gamma), p)|ds \leq |X_t^{(0)}(\gamma)| + (1 + K(\gamma))Ht
\]
\[
\leq |X_t^{(0)}(\gamma)| + (1 + K(\gamma))HT, \forall t \in [0, T]
\]
and
\[
|X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma)| \leq \int_{t_1}^{t_2} |f(s, X_s^{(p)}(\gamma), p)|ds + \int_{t_1}^{t_2} |g(s, X_s^{(p)}(\gamma), p)|dC_s(\gamma)
\]
\[
\leq \int_{t_1}^{t_2} |f(s, X_s^{(p)}(\gamma), p)|ds + K(\gamma) \int_{t_1}^{t_2} |g(s, X_s^{(p)}(\gamma), p)|ds \leq (1 + K(\gamma))H|t_1 - t_2|, \forall t_1, t_2 \in [0, T].
\]

This means that the function set \( \{X_t^{(p)}(\gamma) : p \in (-p_j, p_j)\} \) is both bounded and equicontinuous. We will prove that if \( p \to 0, \) \( X_t^{(p)}(\gamma) - X_t^{(0)}(\gamma) \) uniformly converges to 0 on \([0, T].\)

Let \( \{X_t^{(p_i)}(\gamma)\}_{i=1}^{\infty} \) be an arbitrary subsequence of \( \{X_t^{(p)}(\gamma) : p \in (-p_j, p_j)\} \) with \( p_i \to 0 \) when \( i \to \infty. \) It is clear that \( \{X_t^{(p_i)}(\gamma)\}_{i=1}^{\infty} \) is both bounded and equicontinuous. According to Ascoli-Arzelà Theorem, there exists a subsequence of \( \{X_t^{(p_i)}(\gamma)\}_{i=1}^{\infty} \), simply denoted as \( \{X_t^{(p_i)}(\gamma)\}_{i=1}^{\infty} \), uniformly converges to a function \( Y_t(\gamma), \) i.e., \( Y_t(\gamma) = \lim_{i \to \infty} X_t^{(p_i)}(\gamma). \)

Since the following equation holds
\[
X_t^{(p)}(\gamma) = X_0^{(p)} + \int_0^t f(s, X_s^{(p)}(\gamma), p)ds + \int_0^t g(s, X_s^{(p)}(\gamma), p)dC_s(\gamma),
\]

taking limits on both sides of the above equation obtains
\[
Y_t(\gamma) = X_0^{(p)} + \int_0^t f(s, Y_s(\gamma), 0)ds + \int_0^t g(s, Y_s(\gamma), 0)dC_s(\gamma).
\]

According to (ii) in Condition 2, we have \( Y_t(\gamma) = X_t^{(0)}(\gamma) \) on \([0, T].\)

Since \( \{X_t^{(p_i)}(\gamma)\}_{i=1}^{\infty} \) is an arbitrary subsequence of \( \{X_t^{(p)}(\gamma) : p \in (-p_j, p_j)\}, \) we obtain
\[
\lim_{p \to 0^+ \upharpoonright [0,1]} |X_t^p(\gamma) - X_t^0(\gamma)| = 0. 
\]  
(5)

This completes Lemma 1. \( \square \)

In next lemma, we will prove that for each \( \gamma \in \Gamma \setminus \Gamma_0 \), such \( p_\gamma \) always exists.

**Lemma 2.** For each \( \gamma \in \Gamma \setminus \Gamma_0 \), there exists a positive number \( p_\gamma \leq 1 \) such that \( (t, X_t^p(\gamma)) \in \Delta \) on \([0,T] \times (-p_\gamma, p_\gamma)\).

**Proof.** For \( \gamma \in \Gamma \setminus \Gamma_0 \), set \( \tau = \frac{1}{4H(1 + K(\gamma))} \) and \( N = \lceil \frac{2}{\tau} \rceil + 1 \). Since \( \lim_{p \to 0^+} X_t^0 = X_t^0 \), there exists a \( p_1 > 0 \) such that \( |X_t^0 - X_t^0| < \frac{1}{2} \) when \( p \in [-p_1, p_1] \).

**Step 1.** We will prove that on \([0, \frac{T}{2}]\), \( (t, X_t^p(\gamma)) \in \Delta \) when \( p \in [-p_1, p_1] \). Indeed, on \([0, \frac{T}{2}]\), we have

\[
X_t^p(\gamma) - X_t^0(\gamma) = \int_0^t |f(s, X_s^p(\gamma), p) - f(s, X_s^0(\gamma), 0)| ds + \int_0^t |g(s, X_s^p(\gamma), p) - g(s, X_s^0(\gamma), 0)| ds + K(\gamma).
\]

Assume that there exists a \( t_0 \in [0, \frac{T}{2}] \) such that

\[
\begin{align*}
[X_t^p(\gamma) - X_t^0(\gamma)] &< 1, \quad \forall t \in [0, t_0), \\
[X_t^0(\gamma) - X_t^0(\gamma)] &= 1.
\end{align*}
\]

However, under this assumption, we have

\[
[X_t^p(\gamma) - X_t^0(\gamma)] \leq [X_t^0 - X_t^0] + \int_0^{t_0} |f(s, X_s^p(\gamma), p) - f(s, X_s^0(\gamma), 0)| ds + \int_0^{t_0} |g(s, X_s^p(\gamma), p) - g(s, X_s^0(\gamma), 0)| ds + K(\gamma)
\]

\[
\leq [X_t^0 - X_t^0] + \int_0^{t_0} \left( |f(s, X_s^p(\gamma), p)| + |f(s, X_s^0(\gamma), 0)| \right) ds + K(\gamma)
\]

\[
\leq [X_t^0 - X_t^0] + \int_0^{t_0} 2H_0 s + 2H_0 [1 + 2H0 + K(\gamma)] t_0 \leq \frac{1}{2} + 2H0(1 + K(\gamma))(t_0 \leq \frac{1}{2} + 2H0(1 + K(\gamma)) \tau = 1.
\]

This leads to a contradiction. Thus, on the interval \([0, \frac{T}{2}]\), \( (t, X_t^p(\gamma)) \in \Delta \) when \( p \in [-p_1, p_1] \).

**Step 2.** In Step 1, we proved that there exists a positive number \( p_1 \) such that \( (t, X_t^p(\gamma)) \in \Delta \) on \([0, \frac{T}{2}] \times (-p_1, p_1) \). Moreover, according to Lemma 1, we have

\[
\lim_{p \to 0^+ \upharpoonright [0,\frac{T}{2}]} |X_t^p(\gamma) - X_t^0(\gamma)| = 0.
\]

This means that there exists a positive number \( p_2 \) with \( p_2 \leq p_1 \) such that when \( p \in (-p_2, p_2) \),

\[
[X_t^p(\gamma) - X_t^0(\gamma)] \leq \frac{1}{2}.
\]

Using the same argument as in Step 1, we have on the interval \([0, \frac{T}{2}]\), \( (t, X_t^p(\gamma)) \in \Delta \) when \( p \in (-p_2, p_2) \).

Moreover, since \( p_2 \leq p_1 \), we have on the interval \([0, \frac{T}{2}]\), \( (t, X_t^p(\gamma)) \in \Delta \) when \( p \in (-p_2, p_2) \).

Repeating this process in \( N \) times, we can find a positive number \( p_N \) such that on the interval \([0, T] \), \( (t, X_t^p(\gamma)) \in \Delta \) when \( p \in (-p_N, p_N) \).

Setting \( p_\gamma = p_N \), we complete the proof of Lemma 2. \( \square \)

Theorem 3 largely extends Theorem 1, and much more UDEs are included. In 2012, Gao [5] proved that when the coefficients of UDE satisfy local Lipschitz condition, there exists a unique solution. According to Theorem 3, the solution \( X_t^p \) of uncertain integral Eq. (3) is continuous with respect to \( p \), if \( f(t,x,p) \) and \( g(t,x,p) \) are local Lipschitz continuous.

### 5. Conclusion

This paper presented a basic and a general continuity theorems on solutions of UDEs. In the basic theorem, it requires that the coefficients are global Lipschitz continuous and linear growth, which largely simplifies the proof. However, the global Lipschitz condition is too strict. Then, the general continuity theorem was presented. The condition of general continuity theorem is easy to satisfy, and it has a greater scope of application.

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