Entropy is a measure of the uncertainty associated with a variable whose value cannot be exactly predicated. In uncertainty theory, it has been quantified so far by logarithmic entropy. However, logarithmic entropy sometimes fails to measure the uncertainty. This paper will propose another type of entropy named sine entropy as a supplement, and explore its properties. After that, the maximum entropy principle will be introduced, and the arc-cosine distributed variables will be proved to have the maximum sine entropy with given expected value and variance.

Keywords: uncertain variable, entropy, sine entropy, maximum entropy principle

1. Introduction

Entropy is a criterion to measure the unpredictability of a variable in information sciences. It was initialized by Shannon\textsuperscript{16} for a random variable in 1948. In many real cases, only little information about random variable is available such as expected value and variance, but there are an infinite number of probability distributions satisfying the given information. In this case, Jaynes\textsuperscript{6} suggested to choose the distribution with the maximum entropy, which is called maximum entropy principle.

In 1968, entropy was introduced to fuzzy set by Zadeh\textsuperscript{18} to quantify the fuzziness. However, De Luca and Termini\textsuperscript{3} refined the fuzzy entropy for better use in 1972. Up to now, fuzzy entropy has been studied by many researchers such as Kaufmann\textsuperscript{7}, Yager\textsuperscript{17}, Kosko\textsuperscript{8}, Pal and Pal\textsuperscript{14}, Pal and Bezdek\textsuperscript{15}.

In order to study the uncertainty in human systems, an uncertainty theory was founded by Liu\textsuperscript{9} in 2007 and refined by Liu\textsuperscript{12} in 2010. Based on the uncertain measure, Liu\textsuperscript{9} defined uncertain variable to describe uncertain phenomenon. After that, Liu\textsuperscript{12} provided a definition of logarithmic entropy for uncertain variable. The properties of logarithmic entropy for uncertain variables were investigated by Dai and
Chen\textsuperscript{2}, and the maximum entropy principle for uncertain variables was proposed by Chen and Dai\textsuperscript{1}.

However, logarithmic entropy cannot measure the uncertainty of all the uncertain variables. As a supplement, this paper will propose a new type of entropy for uncertain variables named sine entropy. The rest of this paper is organized as follows. In Section 2, we review some concepts and properties about uncertain variables. The concept of sine entropy is proposed in Section 3 and its properties are explored in Section 4. Then the maximum entropy principle with respect to sine entropy is investigated in Section 5. At last, some remarks are made in Section 6.

2. Preliminary

Uncertainty theory aims at providing a branch of axiomatic mathematics to study human uncertainty in daily life. In contrast to probability theory, uncertainty theory is founded based on uncertain measure with normality axiom, duality axiom, subadditivity axiom, and product axiom. Many researchers have contributed a lot in this area, such as Gao\textsuperscript{4}, Gao, Gao and Ralescu\textsuperscript{5}, Peng and Iwamura\textsuperscript{13}, and Liu\textsuperscript{10}.

In this section, we will introduce some fundamental definitions and results which are indispensable in the rest of this paper.

**Definition 1.** (Liu\textsuperscript{9}) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $M : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $M(\Gamma) = 1$ for the universal set $\Gamma$.

Axiom 2: (Duality Axiom) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$  

Then the triple $(\Gamma, \mathcal{L}, M)$ is called an uncertainty space. Besides, the product uncertain measure on the product $\sigma$-algebra $\mathcal{L}$ was defined by Liu\textsuperscript{11} as follows,

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$.

Then the product uncertain measure $M$ is an uncertain measure satisfying

$$M\left(\bigotimes_{i=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

**Definition 2.** (Liu\textsuperscript{9}) An uncertain variable $\xi$ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.
Definition 3. (Liu\textsuperscript{9}) The uncertainty distribution of an uncertain variable $\xi$ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number $x$.

The inverse function $\Phi^{-1}$ is called an inverse uncertainty distribution of $\xi$. Inverse uncertainty distribution is the most important tool in the arithmetic operation on independent uncertain variables.

Definition 4. (Liu\textsuperscript{11}) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \bigwedge_{i} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Theorem 1. (Liu\textsuperscript{12}) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution

$$\Phi^{-1}(r) = f(\Phi_1^{-1}(r), \cdots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \cdots, \Phi_n^{-1}(1-r)).$$

The expected value is the average of an uncertain variable in the sense of uncertain measure. It is an important index to rank uncertain variables.

Definition 5. (Liu\textsuperscript{9}) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.

Definition 6. (Liu\textsuperscript{9}) Let $\xi$ be an uncertain variable with expected value $e$. Then the variance of $\xi$ is defined by

$$V[\xi] = E[(\xi - e)^2].$$

Generally, different from random variables, the precise variance of an uncertain variable cannot be obtained via its uncertainty distribution only due to the subadditivity of the uncertain measure. In that case, Liu\textsuperscript{9} suggested to take

$$V[\xi] = 2 \int_{e}^{+\infty} (x-e)(1-\Phi(x) + \Phi(2e-x))dx$$

as a stipulation.
Definition 7. (Liu\textsuperscript{11}) Let $\xi$ be an uncertain variable with an uncertainty distribution $\Phi$. Then the logarithmic entropy of $\xi$ is defined by

$$L[\xi] = \int_{-\infty}^{+\infty} H(\Phi(x)) \, dx$$

where $H(t) = -t \ln t - (1-t) \ln(1-t)$.

3. Sine Entropy

In this section, we will propose a concept of sine entropy for uncertain variables as a supplement of logarithmic entropy.

Definition 8. Let $\xi$ be an uncertain variable with an uncertainty distribution $\Phi$. Then the sine entropy of $\xi$ is defined by

$$S[\xi] = \int_{-\infty}^{+\infty} \sin(\pi \Phi(x)) \, dx.$$  

Note that $\sin(\pi x)$ is a symmetric function with respect to $x = 1/2$, and it is strictly increasing in $[0, 1/2]$ and strictly decreasing in $[1/2, 1]$. The sine entropy $\sin(\pi \times M\{A\})$ proposes a good measure to characterize the uncertainty of an event $A$. For an event $A$ with $M\{A\} = 0$, which means $A$ happens almost impossibly, we have $\sin(\pi \times M\{A\}) = 0$. That is, the event $A$ possesses no uncertainty. For an event $A$ with $M\{A\} = 1\textsuperscript{12}$, which means $A$ happens almost surely, we have $\sin(\pi \times M\{A\}) = 0$. That is, the event $A$ possesses no uncertainty, either. For an event $A$ with $M\{A\} = 1/2$, in which case it is the most difficult to predict $A$, we have $\sin(\pi \times M\{A\}) = 1$. That is, the event $A$ possesses the most uncertainty.

Remark 1. Note that logarithmic entropy cannot measure the uncertainty of all uncertain variables. In this case, sine entropy can be employed as a supplement of logarithmic entropy to measure the uncertainty of some uncertain variables. For example, consider a nonnegative uncertain variable $\xi$ with an uncertainty distribution

$$\Phi(x) = 1 - \frac{e}{(x + e) \ln^2(x + e)}, \quad \forall x \geq 0.$$  

Here, $e$ is the base of the natural logarithm. Then we have

$$L[\xi] = \int_{0}^{+\infty} \Phi(x) \ln \Phi(x) - (1 - \Phi(x)) \ln(1 - \Phi(x)) \, dx$$

$$\geq \int_{0}^{+\infty} -(1 - \Phi(x)) \ln(1 - \Phi(x)) \, dx$$

$$= \int_{0}^{+\infty} \frac{-e}{(x + e) \ln^2(x + e)} \ln \left( \frac{e}{(x + e) \ln^2(x + e)} \right) \, dx$$

$$= \int_{0}^{+\infty} \frac{e}{(x + e) \ln(x + e)} + \frac{e \ln(\ln(x + e))}{(x + e) \ln^2(x + e)} - \frac{e}{(x + e) \ln^2(x + e)} \, dx.$$
Since
\[
\int_0^{+\infty} \frac{e}{(x+e)\ln(x+e)} \, dx = +\infty, \quad \int_0^{+\infty} \frac{e \ln (x+e)}{(x+e)\ln^2(x+e)} \, dx < +\infty,
\]
and
\[
\int_0^{+\infty} \frac{e}{(x+e)\ln^2(x+e)} \, dx < +\infty,
\]
we have \( L[\xi] = +\infty \). For the sine entropy,
\[
S[\xi] = \int_0^{+\infty} \sin \left( \pi \left( 1 - \frac{e}{(x+e)\ln^2(x+e)} \right) \right) \, dx
\]
\[
= \int_0^{+\infty} \sin \left( \frac{\pi e}{(x+e)\ln^2(x+e)} \right) \, dx.
\]
Since
\[
\sin \left( \frac{\pi e}{(x+e)\ln^2(x+e)} \right) \sim \frac{\pi e}{(x+e)\ln^2(x+e)}
\]
when \( x \) is large enough, and the right term is integrable on \([0, +\infty)\), we have \( S[\xi] < +\infty \).

**Example 1.** Assume that the uncertain variable \( \xi \) is a constant \( a \) whose uncertainty distribution is
\[
\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x \geq a. \end{cases}
\]
Then it follows from the definition of sine entropy that
\[
S[\xi] = \int_{-\infty}^a \sin(0 \times \pi) \, dx + \int_a^{+\infty} \sin(1 \times \pi) \, dx = 0.
\]
It means a constant has no uncertainty.

**Example 2.** Let \( \xi \) be a linear uncertain variable \( \mathcal{L}(a, b) \) with an uncertainty distribution
\[
\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x > b. \end{cases}
\]
Then it follows from the definition of sine entropy that
\[
S[\xi] = \int_a^b \sin \left( \frac{\pi(x-a)}{b-a} \right) \, dx = \frac{2}{\pi} (b-a).
\]
Example 3. Let $\xi$ be a linear uncertain variable $\mathcal{Z}(a, b, c)$ with an uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < a \\
\frac{x - a}{2(b - a)}, & \text{if } a \leq x \leq b \\
\frac{x + c - 2b}{2(c - b)}, & \text{if } b < x \leq c \\
1, & \text{if } x > c.
\end{cases}
$$

Then it follows from the definition of sine entropy that

$$
S[\xi] = \int_a^b \sin \left( \frac{\pi(x - a)}{2(b - a)} \right) dx + \int_b^c \sin \left( \frac{\pi(x + c - 2b)}{2(c - b)} \right) dx
$$

$$
= \frac{2}{\pi} (b - a) + \frac{2}{\pi} (c - b) = \frac{2}{\pi} (c - a).
$$

Example 4. Let $\xi$ be an arc-cosine uncertain variable $\mathcal{AC}(\varepsilon, \sigma)$ with an uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < \varepsilon - \sqrt{2}\sigma \\
\frac{1}{\pi} \arccos \left( \frac{\sqrt{2}}{2\sigma}(e - x) \right), & \text{if } \varepsilon - \sqrt{2}\sigma \leq x \leq \varepsilon + \sqrt{2}\sigma \\
1, & \text{if } x > \varepsilon + \sqrt{2}\sigma.
\end{cases}
$$

Then it follows from the definition of sine entropy that

$$
S[\xi] = \int_{\varepsilon - \sqrt{2}\sigma}^{\varepsilon + \sqrt{2}\sigma} \sin \left( \pi \times \frac{1}{\pi} \arccos \left( \frac{\sqrt{2}}{2\sigma}(e - x) \right) \right) dx = \frac{\sqrt{2}}{2\pi} \sigma.
$$

4. Properties of Sine Entropy

**Theorem 2.** Let $\xi$ be an uncertainty variable taking values on the interval $[a, b]$. Then

$$
S[\xi] \leq b - a
$$

and the equality holds if $\xi$ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a, b]$.

**Proof.** The theorem follows from the fact that $\sin(t)$ reaches its maximum 1 at $t = 0.5$. \(\square\)

**Theorem 3.** Let $\xi$ be an uncertain variable, and $c$ be a real number. Then

$$
S[\xi + c] = S[\xi].
$$

**Proof.** Let $\Phi$ denote the uncertainty distribution of $\Phi$. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that

$$
S[\xi + c] = \int_{-\infty}^{+\infty} \sin(\pi \Phi(x - c)) dx = \int_{-\infty}^{+\infty} \sin(\pi \Phi(x)) dx = S[\xi].
$$
The theorem is thus verified.

**Theorem 4.** Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then

$$S[\xi] = \pi \int_0^1 \Phi^{-1}(1-r) \cos(\pi r) dr.$$ 

**Proof.** Since $\sin(\pi \Phi(x))$ satisfies

$$\sin(\pi \Phi(x)) = \int_0^{\Phi(x)} \pi \cos(\pi r) dr = -\int_{\Phi(x)}^1 \pi \cos(\pi r) dr,$$

it follows from the definition of sine entropy that

$$S[\xi] = \int_{-\infty}^{+\infty} \sin(\pi \Phi(x)) dx = \int_{-\infty}^0 \sin(\pi \Phi(x)) dx + \int_0^{+\infty} \sin(\pi \Phi(x)) dx$$

$$= \int_0^{+\infty} \Phi^{\Phi(x)} \pi \cos(\pi r) dr - \int_0^{+\infty} \Phi^{-1}(r) \pi \cos(\pi r) dr.$$ 

By Fubini theorem, we have

$$S[\xi] = \int_0^{\Phi(0)} \int_{\Phi^{-1}(r)}^0 \pi \cos(\pi r) dr dr - \int_0^{\Phi(0)} \int_0^{\Phi^{-1}(r)} \pi \cos(\pi r) dr dr$$

$$= -\int_0^{\Phi(0)} \Phi^{-1}(r) \pi \cos(\pi r) dr - \int_0^{\Phi(0)} \Phi^{-1}(r) \pi \cos(\pi r) dr$$

$$= -\pi \int_0^1 \Phi^{-1}(r) \cos(\pi r) dr = \pi \int_0^1 \Phi^{-1}(1-r) \cos(\pi r) dr.$$ 

The theorem is thus verified.

**Example 5.** Let $\xi$ and $\eta$ be independent uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Note that $\xi \eta$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(r) = \Phi^{-1}(r) \Psi^{-1}(1-r)$$

by Theorem 1, we have

$$S[\xi \eta] = \pi \int_0^1 \Upsilon^{-1}(1-r) \cos(\pi r) dr = \pi \int_0^1 \Phi^{-1}(1-r) \Psi^{-1}(1-r) \cos(\pi r) dr.$$ 

**Example 6.** Let $\xi$ and $\eta$ be independent uncertain variables with uncertainty distributions $\Phi$ and $\Psi$, respectively. Since $\xi/\eta$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(r) = \Phi^{-1}(r)/\Psi^{-1}(1-r)$$

by Theorem 1, we have

$$S[\xi/\eta] = \pi \int_0^1 \Upsilon^{-1}(1-r) \cos(\pi r) dr = \pi \int_0^1 \Phi^{-1}(1-r) / \Psi^{-1}(r) \cos(\pi r) dr.$$
Theorem 5. Let $\xi$ and $\eta$ be independent uncertain variables with regular uncertainty distribution $\Phi$ and $\Psi$, respectively. Then

$$S[a\xi + b\eta] = |a|S[\xi] + |b|S[\eta]$$

for any real numbers $a$ and $b$.

Proof. Suppose that $\xi$ and $\eta$ have uncertainty distributions $\Phi$ and $\Psi$, respectively. The theorem will be proved by three steps.

Step 1: We prove $S[a\xi] = |a|S[\xi]$. If $a > 0$, then $a\xi$ has an inverse uncertainty distribution $\Upsilon^{-1}(r) = a\Phi^{-1}(r)$. By Theorem 4, we have

$$S[a\xi] = \pi \int_0^1 a\Phi^{-1}(1-r) \cos(\pi r) dr = a\pi \int_0^1 \Phi^{-1}(1-r) \cos(\pi r) dr = aS[\xi].$$

If $a = 0$, then $S[a\xi] = 0 = aS[\xi]$. If $a < 0$, then $a\xi$ has an inverse uncertainty distribution $\Upsilon^{-1}(r) = a\Phi^{-1}(1-r)$. By Theorem 4, we have

$$S[a\xi] = \pi \int_0^1 a\Phi^{-1}(r) \cos(\pi r) dr = -a\pi \int_0^1 \Phi^{-1}(1-r) \cos(\pi r) dr = -aS[\xi].$$

Thus we have $S[a\xi] = |a|S[\xi]$ for all cases.

Step 2: We prove $S[\xi + \eta] = S[\xi] + S[\eta]$. It follows from Theorem 1 that $\xi + \eta$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(r) = \Phi^{-1}(r) + \Psi^{-1}(r).$$

By Theorem 4, we have

$$S[\xi + \eta] = \pi \int_0^1 (\Phi^{-1}(1-r) + \Psi^{-1}(1-r)) \cos(\pi r) dr = \pi \int_0^1 \Phi^{-1}(1-r) \cos(\pi r) dr + \pi \int_0^1 \Psi^{-1}(1-r) \cos(\pi r) dr = S[\xi] + S[\eta].$$

Step 3: It follows from Step 1 and Step 2 that, for any real numbers $a$ and $b$, we have

$$S[a\xi + b\eta] = |a|S[\xi] + |b|S[\eta].$$

The theorem is thus verified.
5. Maximum Entropy Principle

**Theorem 6.** Let \( \xi \) be an uncertain variable with an expected value \( e \) and variance \( \sigma^2 \). Then

\[
S[\xi] \leq \frac{\sqrt{2}}{2} \pi \sigma
\]

and the equality holds if \( \xi \) is an arc-cosine variable \( \text{AC}(e, \sigma) \).

**Proof.** Let \( \Phi(x) \) be the uncertainty distribution of \( \xi \), and \( \Psi(x) = \Phi(2e - x) \). Then the variance of \( \xi \) is

\[
V[\xi] = 2 \int_{e}^{+\infty} (x - e) (1 - \Phi(x)) dx + 2 \int_{e}^{+\infty} (x - e) \Psi(x) dx = \sigma^2.
\]

Thus there exits a real number \( \rho \) such that

\[
2 \int_{e}^{+\infty} (x - e) (1 - \Phi(x)) dx = \rho \sigma^2, \quad (1)
\]

\[
2 \int_{e}^{+\infty} (x - e) \Psi(x) dx = (1 - \rho) \sigma^2. \quad (2)
\]

Meanwhile, the sine entropy of \( \xi \) is

\[
S[\xi] = \int_{-\infty}^{+\infty} \sin(\pi \Phi(x)) dx = \int_{-\infty}^{e} \sin(\pi \Phi(x)) dx + \int_{e}^{+\infty} \sin(\pi \Phi(x)) dx = \int_{e}^{+\infty} \sin(\pi \Phi(x)) dx + \int_{e}^{+\infty} \sin(\pi \Psi(x)) dx. \quad (3)
\]

Thus the uncertainty distribution \( \Phi \) with maximum sine entropy should maximize \( S[\xi] \) subject to the constraints (1) and (2). The Lagrange multiplier is

\[
L = \int_{e}^{+\infty} \sin(\pi \Psi(x)) dx - \alpha \left( 2 \int_{e}^{+\infty} (x - e) (1 - \Phi(x)) dx - \rho \sigma^2 \right) + \int_{e}^{+\infty} \sin(\pi \Phi(x)) dx - \beta \left( 2 \int_{e}^{+\infty} (x - e) \Psi(x) dx - (1 - \rho) \sigma^2 \right).
\]

By Euler-Lagrange equation, we have

\[
\pi \cos(\pi \Phi(x)) + 2\alpha (x - e) = 0, \quad x \geq e,
\]

\[
\pi \cos(\pi \Psi(x)) - 2\beta (x - e) = 0, \quad x \geq e.
\]

Thus \( \Phi(x) \) and \( \Psi(x) \) have the forms

\[
\Phi(x) = \frac{1}{\pi} \arccos \left( \frac{2\alpha}{\pi} (e - x) \right), \quad x \geq e
\]

\[
\Psi(x) = \frac{1}{\pi} \arccos \left( \frac{2\beta}{\pi} (x - e) \right), \quad x \geq e.
\]
Substituting them into the variance constraints (1) and (2), we get
\[ \alpha = \frac{\pi}{4\sqrt{\rho}}, \quad \beta = \frac{\pi}{4\sqrt{1-\rho}}. \]
Substituting them into the sine entropy (3) of \( \xi \), we obtain that
\[ S[\xi] = \pi\sigma^2 \left( \sqrt{\rho} + \sqrt{1-\rho} \right) \]
which achieves the maximum value when \( \rho = 1/2 \), and the maximum entropy is
\[ S[\xi] = \frac{\sqrt{2}}{2}\pi\sigma. \]
Thus, the maximum entropy distribution with respect to sine entropy is arc-cosine uncertainty distribution \( \mathcal{AC}(e, \sigma) \) when the expect value is \( e \) and the variance is \( \sigma^2 \).

6. Conclusions

This paper proposed a concept of sine entropy for uncertain variable and studied its properties. Besides, maximum entropy principle was employed, and arc-cosine uncertain variable was proved to have a maximum sine entropy given the expected value and variance.

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