Expected Value Model for Optimal Assignment Problem with Uncertain Profits

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Abstract

This paper employs uncertain programming to deal with optimal assignment problem with uncertain profits. Within the framework of uncertain programming, a concept of expected optimal assignment for uncertain optimal assignment problem is proposed, and then an expected value model is constructed. Taking advantage of properties of uncertainty theory, this model can be transformed into a corresponding deterministic form which can be solved by Kuhn-Munkres algorithm.

Keywords: optimal assignment problem, uncertainty theory, uncertain programming, Kuhn-Munkres algorithm

1 Introduction

Optimal assignment problem is usually met by decision maker. Assume that in a company, there are $m$ workers and $n$ jobs ($m \geq n$), and any worker can be assigned to any job. If the profits of the workers in different jobs are different, how can the decision maker find an assignment such that the total profit of the workers is maximized? This problem is called optimal assignment problem (OAP). Usually, the numbers of workers and jobs are equal, i.e., $m = n$. However, the assignment problem can be made rather more flexible than it first appears. Suppose there are four workers and three jobs. Then a fourth dummy job can be invented, perhaps called “sitting still doing nothing”, with a profit of 0 for the worker assigned to it. The assignment problem can then be solved in the usual way.

In early period, assignment problem was investigated in deterministic environment. Many efficient approaches such as Kuhn-Munkres algorithm \cite{15,30}, auction algorithm \cite{1} have been developed. However, in application, some uncertain factors will appear because of the lack of history data, insufficient information or some other reasons. As a result, it is not suitable to employ classical algorithms in these situations.

Some researchers employed probability theory to study optimal assignment problem. The research of random assignment problems dated back to Donath \cite{6}, who investigated the limiting behavior of the linear assignment problem by solving randomly generated instances. Later, Walkup \cite{34} established an upper bound for the expected cost of optimal assignment, which is one of the first results concerning the behavior of random linear assignment problem. In 1982, Burkard and Fincke \cite{3} presented the first asymptotic results for the expected optimal value of random quadratic assignment problem. For more research of random assignment problem, we may consult Pardalos and Ramakrishnan \cite{31}, Kellerer and Wirsching \cite{14}, Linusson and Wåstlund \cite{17}, etc.

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In 1965, Zadeh [39] proposed the concept of fuzzy sets. Since then, many researchers employed fuzzy concepts to assignment problem. For instance, Chen [4] proposed a fuzzy assignment model that did not consider the differences of individuals, and also proved some theorems. In 1994, Herrera et al. [11] investigated the fuzzy matching problem using linguistic variables. Lin and Wen [16] proposed an efficient algorithm based on the labeling method for the fuzzy assignment problem. Many other researchers, such as Feng and Yang [7], Majumdar and Bhunia [29], Ye and Xu [38], Liu and Gao [18], Kagade and Bajaj [13], have done a lot of work in this field.

However, if uncertain factor comes from the decision maker's empirical estimation, it is not suitable to employ random variable or fuzzy variable to describe the uncertain factor. Fortunately, in order to deal with human data, such as empirical estimation, uncertainty theory was proposed by Liu [19] in 2007. Liu [27] said that “When the sample size is too small (even no-sample) to estimate the probability distribution, some domain experts are invited to evaluate the belief degree of each event. In this situation, the belief degree usually has much larger variance than the long-run frequency and we should deal with it by uncertainty theory.” It is too adventurous if we deal with the belief degree by probability theory, because it may lead to counterintuitive results.

This paper concerns about uncertain optimal assignment problem (UOAP), in which profit is uncertain. We regard profit as uncertain variable. In order to construct an uncertain model, we first introduce the concept of expected optimal assignment for optimal assignment problem with uncertain profits. After that, we construct the expected value model for uncertain optimal assignment problem. Under the framework of uncertainty theory, the model can be transformed into a corresponding deterministic form. This provides an effective method to find a solution to the expected value model, that is, we can employ any classical algorithm to find an optimal assignment on the deterministic problem. In this paper, we employ Kuhn-Munkres algorithm.

The paper is organized as follows: Section 2 presents some basic concepts and properties selected from uncertainty theory, and then describes the history and application of uncertain programming. In Section 3, uncertain optimal assignment problem is described, then the concept of expected optimal assignment is proposed. Section 4 constructs the expected value model and derives the method to find expected optimal assignment for uncertain optimal assignment problem. Section 5 illustrates the proposed method by an example. The last section concludes this paper with a brief summary.

2 Preliminaries

In order to model human language like “about 50km”, “low speed”, and “big size”, uncertainty theory was founded by Liu [19] in 2007. Recently, much research work has been done on the development of uncertainty theory. For example, uncertain calculus was initialized by Liu [21] to deal with differentiation and integration of functions of uncertain processes. Based on uncertain calculus, Liu [20] introduced uncertain differential equation. After that, Chen and Liu [5] proved the existence and uniqueness theorem for uncertain differential equations. Furthermore, uncertain inference was introduced by Liu [21] via conditional uncertain measure. Later, Gao et al. [8] derived some expressions of Liu’s inference rule for uncertain systems. In addition, uncertainty theory was also applied to uncertain statistics (Liu [25], Wang et al. [35], [36]), uncertain risk analysis and uncertain reliability analysis (Liu [24]), uncertain logic (Liu [26]), uncertain finance (Liu [21], Peng and Yao [32]), and uncertain control (Liu [23], Zhu [40]). To explore the recent developments of uncertainty theory, the readers may consult [25].

2.1 Uncertainty Theory

In this section, we introduce some concepts and results from uncertainty theory, which will be used throughout in this paper.
Let $\Gamma$ be a nonempty set, $\mathcal{L}$ a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is called an event. For any $\Lambda \in \mathcal{L}$, a set function $M\{\cdot\}$ is said to be an uncertain measure if it satisfies the following three axioms [19]:

1. **(Normality Axiom)** $M\{\Gamma\} = 1$;
2. **(Duality Axiom)** $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
3. **(Subadditivity Axiom)** For every countable sequence of events $\{\Lambda_i\}$, we have
   \[ M\left(\bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \]

The triplet $(\Gamma, \mathcal{L}, M)$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, the fourth axiom called product axiom was presented by Liu [21].

4. **(Product Axiom)** Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots, n$. Then the product uncertain measure $M$ is an uncertain measure on the product $\sigma$-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying
   \[ M\left(\prod_{k=1}^{n} \Lambda_k \right) = \min_{1 \leq k \leq n} M_k\{\Lambda_k\} \]

where $\Lambda_k \in \mathcal{L}_k$, for $k = 1, 2, \cdots, n$.

**Definition 2.1** [19] An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set
   \[ \{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\} \]

is an event.

**Definition 2.2** [19] The uncertainty distribution $\Phi : \mathbb{R} \rightarrow [0, 1]$ of an uncertain variable $\xi$ is defined by
   \[ \Phi(x) = M\{\gamma \in \Gamma | \xi(\gamma) \leq x\} \]

for any real number $x$. The inverse function $\Phi^{-1}$ is called the inverse uncertainty distribution of $\xi$.

**Example 2.1:** The zigzag uncertain variable $\xi = Z(a, b, c)$ has an uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
\frac{x - a}{2(b - a)}, & \text{if } a \leq x \leq b \\
\frac{x + c - 2b}{2(c - b)}, & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c.
\end{cases}
\]

**Definition 2.3** [19] Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

\[
E[\xi] = \int_{-\infty}^{+\infty} M\{\xi \geq r\} \, dr - \int_{-\infty}^{0} M\{\xi \leq r\} \, dr
\]

provided that at least one of the two integrals is finite.

If $\xi$ has an uncertainty distribution $\Phi$, then the expected value may be calculated by

\[
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x)) \, dx - \int_{-\infty}^{0} \Phi(x) \, dx.
\]
Theorem 2.1 [25] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Example 2.2: Let $\xi_1$ and $\xi_2$ be independent uncertain variables with uncertainty distributions $\Phi_1$ and $\Phi_2$, respectively. Since the function

$$f(x_1, x_2) = x_1 - x_2$$

is strictly increasing with respect to $x_1$ and strictly decreasing with respect to $x_2$, the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha).$$

Example 2.3: Let $\xi_1, \xi_2, \xi_3$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \Phi_3$, respectively. Since the function

$$f(x_1, x_2) = (x_1 + x_2)/x_3$$

is strictly increasing with respect to $x_1, x_2$ and strictly decreasing with respect to $x_3$, the inverse uncertainty distribution of $(x_1 + x_2)/x_3$ is

$$\Psi^{-1}(\alpha) = (\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha))/\Phi_3^{-1}(1-\alpha).$$

Furthermore, assume the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$. It has been proved by Liu and Ha [28] that the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))d\alpha.$$
value subject to a set of chance constraints, Liu [22] proposed the following uncertain programming model,

\[
\begin{align*}
\min_x & \quad E[f(x, \xi)] \\
\text{subject to :} & \quad M\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \ldots, p.
\end{align*}
\]

A key problem in the research area of uncertain programming is how to solve the model like (2.1). Fortunately, under the framework of uncertainty theory, the uncertain programming model (2.1) is equivalent to the crisp model

\[
\begin{align*}
\min_x & \quad \int_0^1 f(x, \Phi^{-1}_1(\alpha), \ldots, \Phi^{-1}_m(\alpha), \Phi^{-1}_{m+1}(1-\alpha), \ldots, \Phi^{-1}_n(1-\alpha))d\alpha \\
\text{subject to :} & \quad g_j(x, \Phi^{-1}_1(\alpha_j), \ldots, \Phi^{-1}_k(\alpha_j), \Phi^{-1}_{k+1}(1-\alpha_j), \ldots, \Phi^{-1}_n(1-\alpha_j)) \leq 0, \quad j = 1, 2, \ldots, p.
\end{align*}
\]

where \(f(x, \xi_1, \xi_2, \ldots, \xi_n)\) is strictly increasing with respect to \(\xi_1, \xi_2, \ldots, \xi_m\) and strictly decreasing with respect to \(\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n\), and \(g_j(x, \xi_1, \xi_2, \ldots, \xi_n)\) is strictly increasing with respect to \(\xi_1, \xi_2, \ldots, \xi_k\) and strictly decreasing with respect to \(\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n\).

After converting the uncertain programming model to a crisp model, we may solve it by simplex method, branch-and-bound method, genetic algorithm, neural networks, tabu search, and so on.

In order to show the applications of the model as mentioned above, the models of uncertain programming on uncertain system reliability design, uncertain machine scheduling problem, uncertain vehicle routing problem, uncertain project scheduling problem and so on, were given by Liu [22], respectively. In addition, Bhattacharyya et al. [2] introduced the concept of multiple objective uncertain optimization problems. In 2009, Yan [37] provided two models for portfolio selection in which the securities are assumed to be uncertain variables. After that, Huang [12] introduced a risk curve and developed a mean-risk model for uncertain portfolio selection. Furthermore, two models of economic order quantity for inventory based on uncertain theory was provided by Rong [33]. In addition, Gao [9] investigated the shortest problem with uncertain arc lengths, and given the \(\alpha\)-shortest path model and the most shortest path model in an uncertain network, and Gao [10] proposed two uncertain models for single facility location problem.

3 Problem Description

An optimal assignment problem with \(n\) workers and \(n\) jobs can be represented in a profit matrix \(P\), where \(p_{ij}\) indicate the profit made by worker \(i\) on job \(j\). The problem is to find an assignment that maximizes the total profit of the workers,

\[
P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix}.
\]

In classical deterministic optimal assignment problem, each profit \(p_{ij}\) is a positive crisp value. Many effective algorithm can be employed to solve the deterministic optimal problem, such as Kuhn-Munkres Algorithm.

Following example illustrates how to obtain the optimal assignment by Kuhn-Munkres algorithm.

Example 3.1: Assume there are four workers and four jobs, and the profit matrix is
After employing of the Kuhn-Munkres algorithm to the profit matrix $P$, an optimal assignment $A$ is presented in Table 1,

Table 1: An Optimal Assignment $A$

<table>
<thead>
<tr>
<th>worker</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>job</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>profit</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

whose total profit of the workers is $4 + 3 + 5 + 6 = 18$.

However, in practical, because of the lack of history data, some uncertain factors will appear. In this situation, the profit data can only be obtained from the decision maker’s empirical estimation. Then how can we deal with these uncertain factors? Fortunately, uncertainty theory provides a new tool to deal with uncertain information, especially expert data and subjective estimation.

In this paper, the uncertain factor for optimal assignment problem is the profits for workers. We employ uncertain variable to describe the profit of each worker, that is, each profit $p_{ij}$ is replaced by a nonnegative uncertain variable $\xi_{ij}$. Before starting model construction, some notations and assumptions are listed in Table 2.

Table 2: List of Notations and Assumptions

| $\xi_{ij}$ | uncertain variable of the profit made by worker $i$ on job $j$, and all the uncertain variables $\xi_{ij}$ are positive and independent |
| $\tilde{P}$ | profit matrix with uncertain profit $\xi_{ij}$ |
| $x_{ij}$ | zero and one decision variable on profit $\xi_{ij}$ |
| $\tilde{A}$ | an assignment for $\tilde{P}$ |

Clearly, $\tilde{A}$ is an assignment if and only if

\[
\begin{aligned}
\sum_{1 \leq j \leq n} x_{ij} &= 1 \quad \text{for } i = 1, 2, \cdots, n \\
\sum_{1 \leq i \leq n} x_{ij} &= 1 \quad \text{for } j = 1, 2, \cdots, n \\
x_{ij} &= \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{aligned}
\]

Remark 1: $x_{ij} = 0$ means that worker $i$ is not assigned to job $j$; $x_{ij} = 1$ means that worker $i$ is assigned to job $j$. The first constraint requires that every worker is assigned to exactly one job, and the second constraint requires that every job is assigned exactly one worker.

Different from that in a deterministic profit matrix $P$, the optimal assignment profit is an uncertain variable in an uncertain profit $\tilde{P}$. Then, for uncertain profit matrix $\tilde{P}$, how can we obtain the optimal assignment?

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable. In order to obtain the decision with maximum expected return, we give the concept of expected optimal assignment for uncertain optimal assignment problem.
Definition 3.1 Let $\tilde{P}$ be an uncertain profit matrix, $\tilde{A}^*$ an assignment. Then $\tilde{A}^*$ is called expected optimal assignment (EOA) if

$$E[w(\tilde{A}^*)] \geq E[w(\tilde{A})]$$

holds for assignment $\tilde{A}$, where $w(\tilde{A}^*)$ stands for the total profit of expected optimal assignment $\tilde{A}^*$ and $w(\tilde{A})$ stands for the total profit of assignment $\tilde{A}$.

As is mentioned before, the optimal assignment profit is an uncertain variable in uncertain profit matrix $\tilde{P}$. How can we get expected optimal assignment? In Section 4, we will give the answer.

4 Expected Value Model

Within the framework of uncertainty theory, the concept of expected optimal assignment in uncertain phenomena is proposed in above section.

In this section, we will construct expected valued model by the concept of expected optimal assignment. Fortunately, taking advantage of some properties of uncertainty theory, this model can be transformed into a corresponding deterministic form.

Theorem 4.1 Let $\tilde{P}$ be an uncertain profit matrix. Then, the expected optimal assignment of $\tilde{P}$ is just the optimal assignment of $P$, where $P$ a profit matrix, and $p_{ij} = E[\xi_{ij}]$.

Proof According to Definition 3.1, the expected value model can be presented as following:

$$\begin{aligned}
\max \quad & E\left[ \sum_{0 \leq i, j \leq n} x_{ij} \xi_{ij} \right] \\
\text{s.t.} \quad & \sum_{1 \leq j \leq n} x_{ij} = 1 \quad \text{for } i = 1, 2, \cdots, n \\
& \sum_{1 \leq i \leq n} x_{ij} = 1 \quad \text{for } j = 1, 2, \cdots, n \\
& x_{ij} = \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{aligned}$$  \hspace{1cm} (4.1)

Note that, $\xi_{ij}, i, j = 1, 2, \cdots, n$, are independent, by Theorem 2.2, the model (4.1) can be equivalently transformed to the following deterministic model:

$$\begin{aligned}
\max \quad & \sum_{0 \leq i, j \leq n} x_{ij} E[\xi_{ij}] \\
\text{s.t.} \quad & \sum_{1 \leq j \leq n} x_{ij} = 1 \quad \text{for } i = 1, 2, \cdots, n \\
& \sum_{1 \leq i \leq n} x_{ij} = 1 \quad \text{for } j = 1, 2, \cdots, n \\
& x_{ij} = \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{aligned}$$  \hspace{1cm} (4.2)

In fact, the model (4.2) describes the optimal assignment in profit matrix $P$, whose elements $p_{ij}$ are $E[\xi_{ij}]$. The theorem is proved.

Example 4.1 Suppose the elements of uncertain profit matrix $\tilde{P}$ are all linear uncertain variables $\xi_{ij} \sim \mathcal{L}(a_{ij}, b_{ij}), i, j = 1, 2, \cdots, n$, respectively. Then the expected value model can be transformed into the following form:

$$\begin{aligned}
\max \quad & \frac{1}{2}(a_{ij} + b_{ij}) x_{ij} \\
\text{s.t.} \quad & \sum_{1 \leq j \leq n} x_{ij} = 1 \quad \text{for } i = 1, 2, \cdots, n \\
& \sum_{1 \leq i \leq n} x_{ij} = 1 \quad \text{for } j = 1, 2, \cdots, n \\
& x_{ij} = \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{aligned}$$  \hspace{1cm} (4.3)
Example 4.2 Suppose the elements of uncertain profit matrix $\tilde{P}$ are all zigzag uncertain variables $\xi_{ij} \sim Z(a_{ij}, b_{ij}, c_{ij})$, $i, j = 1, 2, \cdots, n$, respectively. Then the expected value model can be transformed into the following form:

$$\begin{align*}
\max & \quad \sum_{0\leq i,j \leq n} \frac{1}{4}(a_{ij} + 2b_{ij} + c_{ij})x_{ij} \\
\text{s.t.} & \quad \sum_{1 \leq j \leq n} x_{ij} = 1 \quad \text{for } i = 1, 2, \cdots, n \\
& \quad \sum_{1 \leq i \leq n} x_{ij} = 1 \quad \text{for } j = 1, 2, \cdots, n \\
& \quad x_{ij} = \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{align*}$$

(4.4)

Example 4.3 Suppose the elements of uncertain profit matrix $\tilde{P}$ are all normal uncertain variables $\xi_{ij} \sim N(\epsilon_{ij}, \sigma_{ij})$, $i, j = 1, 2, \cdots, n$, respectively. Then the expected value model can be transformed into the following form:

$$\begin{align*}
\max & \quad \sum_{0\leq i,j \leq n} \epsilon_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{1 \leq j \leq n} x_{ij} = 1 \quad \text{for } i = 1, 2, \cdots, n \\
& \quad \sum_{1 \leq i \leq n} x_{ij} = 1 \quad \text{for } j = 1, 2, \cdots, n \\
& \quad x_{ij} = \{0, 1\} \quad \text{for } i, j = 1, 2, \cdots, n.
\end{align*}$$

(4.5)

5 Numerical Experiment

In the past, the methods of genetic algorithm, tabu search algorithm or some other Hybrid intelligent algorithms were usually employed to find the optimal solution to uncertain programming problems. However, the rigor of these methods are not good. Theorem 4.1 provides an effective method to obtain expected optimal assignment in uncertain profit matrix $\tilde{P}$. That is, we only need to employ Kuhn-Munkres algorithm to find the optimal assignment of a corresponding deterministic profit matrix $P$.

Roughly speaking, the method to obtain expected optimal assignment in uncertain profit matrix $\tilde{P}$ can be summarized as follows:

**Step 1:** Calculate $E[\xi_{ij}]$, for each element $\xi_{ij}$ in $\tilde{P}$;

**Step 2:** Construct a deterministic profit matrix $P$, whose elements $p_{ij} = E[\xi_{ij}]$;

**Step 3:** Employ Kuhn-Munkres algorithm to get the optimal assignment in profit matrix $P$.

The optimal assignment we get in Step 3 is just the expected optimal assignment in uncertain profit matrix $\tilde{P}$.

In this section, we give an example to illustrate the method as mentioned above. For the convenience of description, we summarize the problem as follows. Suppose that a taxi firm has five drivers available, and five customers wishing to be picked up as soon as possible. Now, the task for the decision maker is to make the assignment plan such that the total profit (or time) of the drivers is maximized (or minimized). At the beginning of the task, the decision maker needs to obtain the basic data, such as level of drivers’ understanding on the traffic condition, driving skill of each driver, and so on. In fact, since the plan is made in advance, we usually cannot obtain these data exactly. In this situation, we usually obtain the uncertain data by means of expert advice. Assume that all uncertain variables $\xi_{ij}$ are listed in Table 3.

<table>
<thead>
<tr>
<th>Table 3: List of $\xi_{ij}$</th>
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<td>$\xi_{ij}$</td>
</tr>
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8
\[ \xi_{ij} \text{ value of } \xi_{ij} \quad \xi_{ij} \text{ value of } \xi_{ij} \]
\[
\begin{array}{cccc}
\xi_{11} & L(2,4) & \xi_{34} & Z(3,5,6) \\
\xi_{12} & Z(3,4,5,5) & \xi_{35} & Z(1,3,5) \\
\xi_{13} & Z(2,3,4) & \xi_{41} & L(5,7) \\
\xi_{14} & Z(2,3,5) & \xi_{42} & Z(4,4,5,5) \\
\xi_{15} & N(4,1) & \xi_{43} & N(5,2) \\
\xi_{21} & L(1,5,4,5) & \xi_{44} & N(6,1) \\
\xi_{22} & Z(2,3,5) & \xi_{45} & Z(3,4,5) \\
\xi_{23} & N(2,1) & \xi_{51} & Z(2,3,5) \\
\xi_{24} & Z(4,5,5,6) & \xi_{52} & L(1,5) \\
\xi_{25} & N(3,2) & \xi_{53} & Z(3,3,5,5) \\
\xi_{31} & Z(3,4,6) & \xi_{54} & L(2,5,5,5) \\
\xi_{32} & Z(2,3,5) & \xi_{55} & Z(2,3,4) \\
\xi_{33} & L(3,5) & - & - \\
\end{array}
\]

**Step 1:** Calculate \( E[\xi_{ij}] \) of \( \xi_{ij} \) listed in Table 4.

<table>
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<tr>
<th>( \xi_{ij} )</th>
<th>( E[\xi_{ij}] )</th>
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<td>( \xi_{55} )</td>
<td>3</td>
</tr>
<tr>
<td>( \xi_{33} )</td>
<td>4</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Step 2:** From the data in Table 4, we construct a deterministic profit matrix

\[
P = \begin{pmatrix}
3 & 4.25 & 3 & 3.25 & 4 \\
3 & 3.25 & 2 & 5.25 & 3 \\
4.25 & 3.25 & 4 & 4.75 & 3 \\
6 & 4.5 & 5 & 6 & 4 \\
3.25 & 3 & 3.75 & 4 & 3 \\
\end{pmatrix}
\]

**Step 3:** Employ Kuhn-Munkres algorithm to the profit matrix \( P \), the optimal assignment \( A \) is presented in Table 5,

<table>
<thead>
<tr>
<th>worker</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>job</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>profit</td>
<td>4.25</td>
<td>5.25</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

9
whose total profit of workers is \( 4.25 + 5.25 + 4 + 6 + 3 = 22.5 \).

According to Theorem 4.1, the expected optimal assignment in uncertain profit matrix \( \tilde{P} \) is presented in Table 6,

<table>
<thead>
<tr>
<th>worker</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>job</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>profit</td>
<td>4.25</td>
<td>5.25</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

whose total profit of workers is \( 4.25 + 5.25 + 4 + 6 + 3 = 22.5 \).

This means that in order to obtain the decision with maximum expected return, the decision maker should assign his workers according to the optimal. The corresponding maximum profit is 22.5.

6 Conclusion

In practical application, some uncertain factors often appear in optimal assignment problem. This paper concerns about uncertain optimal assignment problem, in which profit is uncertain. The uncertain factor is described by uncertainty theory. In order to construct uncertain model, the concept of expected optimal assignment is proposed. After that, the expected value model for uncertain optimal assignment problem is proposed. With the help of uncertainty theory, the expected value model can be transformed into a corresponding deterministic form, and then we can use Kuhn-Munkres algorithm to find its solution. At last, an example is given to illustrate the method.

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References


Appendix

The Kuhn-Munkres algorithm is easier to describe if we formulate the problem using a bipartite graph. Given a weighted complete bipartite graph $G = (X, Y)$ where edge $xy$ has weight $w(xy)$, find a matching $H$ from $X$ to $Y$ with maximum weight. In an application, $X$ could be a set of workers, $Y$ could be a set of jobs, and $w(xy)$ could be the profit made by assigning worker $x$ to job $y$.

A feasible vertex labeling in $G$ is a real-valued function $l$ on $X \cup Y$ such that, for all $x \in X$ and $y \in Y$,

$$l(x) + l(y) \geq w(xy).$$

If $l$ is a feasible vertex labeling, we denote by $G_l$ the subgraph of $G$ which contains those edges where $l(x) + l(y) = w(xy)$.  


The Kuhn-Munkres Algorithm

Start with an arbitrary feasible vertex labeling \( l \), then determine \( G_l \) and choose an arbitrary matching \( H \) in \( G_l \).

**Step 1:** If \( H \) is complete for \( G \), then \( H \) is optimal. Stop. Otherwise, there is some unmatched \( x \in X \). Set \( S = \{ x \} \) and \( T = \emptyset \).

**Step 2:** If \( N_{G_l}(S) \neq T \), go to Step 3. Otherwise, \( N_{G_l}(S) = T \). Compute

\[
\alpha_l = \min_{x \in S, y \in T} \{ l(x) + l(y) - w(xy) \}
\]

and construct a new labeling \( \hat{l} \) by

\[
\hat{l}(v) = \begin{cases} 
  l(v) - \alpha_l & \text{for } v \in S \\
  l(v) + \alpha_l & \text{for } v \in T \\
  l(v) & \text{otherwise}
\end{cases}
\]

Note that \( \alpha_l > 0 \) and \( N_{G_l}(S) \neq T \). Replace \( l \) by \( \hat{l} \) and \( G_l \) by \( G_{\hat{l}} \).

**Step 3:** Choose a vertex \( y \) in \( N_{G_l}(S) \), not in \( T \). If \( y \) is matched in \( H \), say with \( z \in X \), replace \( S \) by \( S \cup \{ z \} \) and \( T \) by \( T \cup \{ y \} \), and go to Step 2. Otherwise, there will be an \( H \)-augmenting path \( P \) from \( x \) to \( y \). Replace \( H \) by \( \hat{H} = H \Delta E(P) \) and go to Step 1.

**Remark 2:** In Step 2, the number of computations required to compute \( G_{\hat{l}} \) is clearly of order \( n^2 \). Since the algorithm can cycle through Step 2 and Step 3 at most \( |X| \) times before finding an \( H \)-augmenting path, and since the initial matching can be augmented at most \( |X| \) times before an optimal matching is found, we see that the Kuhn-Munkres algorithm is an effective algorithm.