Entropy Operator for Membership Function of Uncertain Set

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Abstract

Similar to fuzzy set on a possibility space, uncertain set is a set-valued function on an uncertainty space, and attempts to model unsharp concepts. Entropy provides a quantitative measurement of the uncertainty associated with an uncertain set. This paper presents a formula for calculating the entropy of an uncertain set via its inverse membership function. Based on the formula, the entropy operator is shown to satisfy positive linearity property. In addition, this paper proposes a concept of relative entropy to describe the divergence between the membership functions of two uncertain sets.

Keywords: uncertain set; membership function; entropy; relative entropy

1 Introduction

Entropy is used to measure the degree of possibility associated with an indeterminacy phenomena. It was first presented by Shannon [32] for a random variable in 1949. In order to model the divergence between two random variables, a concept of relative entropy was proposed by Kullback and Leibler [17] in 1951. As we know, when the expected value and variance of a random variable are given, we can find many probability distributions for the random variable. In 1957, Jaynes [13] suggested to choose the one with the maximum entropy as the probability distribution of the random variable in practice, that is the maximum entropy principle nowadays.

Although probability has been used to model indeterminacy phenomena for a long time, due to the incompleteness of the available information, sometimes it is difficult to get the precise probability distribution of an indeterminacy quantity. In this case, we have to rely on the experts' belief degree that each event will occur. So far, many theories has been proposed to deal with the belief degree, such as fuzzy theory (Zadeh [40]), Dempster-Shafer theory (Dempster [7], Shafer [31]), and rough theory (Pawlak [29]). Entropy was first introduced to fuzzy set by Zadeh [41] to quantify the fuzziness in 1968, in which he defined the entropy of a fuzzy set as a weighted Shannon entropy. After that, based on four requirements that an entropy should satisfy, De Luca and Termini [12] refined the fuzzy entropy in form of functions of Shannon entropy. In 1975, Kaufmann [14] attempted to measure the fuzziness of a fuzzy set via the distance between the fuzzy set and its closest classic set. After that, Yager [34, 35] studied the fuzziness contained in a fuzzy proposition. In 1986, Kosko [16] employed fuzzy entropy to measure the information in a fuzzy set. For a survey of fuzzy entropy, please refer to Pal and Bezdek [28].

The concept of entropy in the framework of Dempster-Shafer theory was first proposed by Yager [36] in 1983. Then in 1987, Yager [37] discussed the entropy of belief structure. After that, Klir and Ramer [15] and Harmanec and Klir [11] proposed a measure of discord to describe the uncertain evidence, and discussed its main properties. An application of entropy in Dempster-Shafer theory to multi-sensor information fusion was given by Basir and Yuan [1] in 2007. In 1998, Düntsch and Gediga [8] and Beaubouef *et al.* [2] proposed two different methods to measure the uncertainty of rough sets. Then Liang *et al.* [18, 19] defined conditional entropy and rough entropy for the rough information. An application of rough entropy to feature selection and recognition was given by Swiniarski and Skowron [33] in 2003.

In 2007, an uncertainty theory was founded by Liu [20] to deal with the uncertainty associated with the experts' belief degree. Then in 2010, Liu [22] refined the uncertainty theory based on normality, duality, subadditivity and product axioms. In order to model the uncertain quantities, Liu [20] proposed a concept of uncertain variable. Then Peng and Iwamura [27] gave a sufficient and necessary condition for a function being the uncertainty distribution of an uncertain variable. You [39] discussed the convergence of a sequence of uncertain variables. Qin and Kar [30] gave an application in inventory model. In 2009, Liu [21] provided a concept of entropy for uncertain variable in the form of logarithm function. The maximum entropy principle for uncertain variable was proposed by Chen and Dai [4] in 2011, and the properties of entropy for uncertain variables were investigated by Dai and Chen [6] in 2012. In order to measure the difference between two uncertain variables, Chen *et al.* [5] introduced relative entropy to uncertain variables. In addition, Yao *et al.* [38] proposed entropy in the form of sine function for uncertain variables.

Uncertain set, as a generalization of uncertain variable, was proposed by Liu [23] in 2010 as a set-valued function on an uncertainty space. Inspired by fuzzy set, a membership function was defined to describe the uncertain measure that a real number belongs to an uncertain set. In 2010, Gao *et al.* [9] proposed some inference rules with multiple antecedents based on uncertain sets, which was further applied to balance the inverted pendulum by Gao [10]. Besides, Liu [24] applied the uncertain sets to the field of linguistic summarizer. In 2012, Liu [25] recast the concept of uncertain set.

The concept of entropy for uncertain sets was proposed by Liu [24] in 2011. In this paper, we will give a formula to calculate the entropy of uncertain set via inverse membership function, based on which we will also verify the positive linearity of the entropy operator. In addition, we will propose a concept of relative entropy to describe the divergence between the membership functions of two uncertain sets. The rest of this paper is structured as follows. The next section intends to introduce some concepts about uncertain set. After that, we introduce the entropy for uncertain set in Section 3. In section 4, we provide a formula for calculating the entropy of an uncertain set via inverse membership function, based on which we verify the positive linearity of entropy in Section 5. The relative entropy for uncertain sets is presented in Section 6. At last, some remarks are made in Section 7.

2 Preliminary

This section will introduce some basic concepts and theorems about uncertain set that will used throughout the paper. In order to provide a quantitative measurement that an uncertain phenomenon will occur, an uncertain measure was defined as below.

Definition 1. *(Liu [20]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. The product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by Liu [21] as follows,
Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra satisfies

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

For example, assume $\lambda(x)$ is a nonnegative function on the real number set $\Re$ such that

$$\sup_{x \neq y}(\lambda(x) + \lambda(y)) = 1.$$

For each event Λ in the Borel set $\mathcal{B}$, the set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x \in \Lambda} \lambda(x), & \text{if } \sup\limits_{x \in \Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x \in \Lambda^c} \lambda(x), & \text{if } \sup\limits_{x \in \Lambda} \lambda(x) \geq 0.5 \end{cases}$$

is an uncertain measure, and $(\Re, \mathcal{B}, \mathcal{M})$ is an uncertainty space. Essentially, an uncertain set is a set-valued function on an uncertainty space.

Definition 2. *(Liu [25]) An uncertainty set is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers, i.e., for any Borel set B of real numbers, the following two sets*

$$\{\xi \subset B\} = \{\gamma \in \Gamma \,|\, \xi(\gamma) \subset B\},$$
$$\{B \subset \xi\} = \{\gamma \in \Gamma \,|\, B \subset \xi(\gamma)\}$$

are events.

Definition 3. *(Liu [25]) An uncertain set ξ is said to have a membership function μ if the equations*

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x),$$
$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x)$$

hold for any Borel set B of real numbers.

For example, let the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0,1]$ with $\mathcal{M}\{[0,\gamma]\} = \gamma$. For each $\gamma \in [0,1]$, define $\xi(\gamma) = [-\sqrt{1-\gamma}, \sqrt{1-\gamma}]$. Then ξ is an uncertain set with a membership function

$$\mu(x) = \begin{cases} 1 - x^2, & \text{if } x \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

An uncertain set ξ is called a triangular uncertain set if it has a triangular membership function

$$\mu(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ \dfrac{x-c}{b-c}, & \text{if } b \le x \le c \\ 0, & \text{otherwise} \end{cases}$$

denoted by (a, b, c), and an uncertain set η is called a trapezoidal uncertain set if it has a trapezoidal membership function

$$\nu(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } b \le x \le c \\ \dfrac{x-d}{c-d}, & \text{if } c \le x \le d \\ 0, & \text{otherwise} \end{cases}$$

denoted by (a, b, c, d). Liu [23] proved that a real-valued function μ is a membership function if and only if $0 \le \mu(x) \le 1$. A membership function is said to be regular if there exists a point x_0 such that $\mu(x_0) = 1$ and $\mu(x)$ is unimodal about the point x_0.

Definition 4. *(Liu [25]) Let ξ be an uncertain set with a membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \{x \in \Re \,|\, \mu(x) \ge \alpha\}, \quad \forall \alpha \in [0,1]$$

is called the inverse membership function of ξ.

If μ is a regular membership function, then the function $\mu_l^{-1}(\alpha) = \inf \mu^{-1}(\alpha)$ is called the left inverse membership function, and the function $\mu_r^{-1}(\alpha) = \sup \mu^{-1}(\alpha)$ is called the right inverse membership function. Note that $\mu_l^{-1}(\alpha)$ is increasing and $\mu_r^{-1}(\alpha)$ is decreasing with respect to α. The left and right inverse membership functions of a triangular uncertain set (a, b, c) are

$$\mu_l^{-1}(\alpha) = a + (b-a)\alpha, \quad \mu_r^{-1}(\alpha) = c + (b-c)\alpha,$$

respectively, and the left and right inverse membership functions of a trapezoidal uncertain set (a, b, c, d) are

$$\mu_l^{-1}(\alpha) = a + (b-a)\alpha, \quad \mu_r^{-1}(\alpha) = d + (c-d)\alpha,$$

respectively.

Definition 5. *(Liu [26]) The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^* \in B_i\}, \quad \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^* \in B_i\},$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}, i = 1, 2, \cdots, n$, respectively.

For regular independent uncertain sets ξ and η with membership functions $\mu(x)$ and $\nu(x)$, respectively, Liu [25] proved that the union $\xi \cup \eta$ has a membership function $\mu(x) \vee \nu(x)$, the intersection $\xi \cap \eta$ has a membership function $\mu(x) \wedge \nu(x)$, and the complement ξ^c has a membership function $1 - \mu(x)$. For the arithmetic operations of the uncertain sets, Liu [25] provided the following theorem via inverse membership functions, that will be used to verify the positive linearity in this paper.

Theorem 1. *(Liu [25]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with regular inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \cdots, \mu_n^{-1}$, respectively. If the function f is increasing with respect to $x_1, x_2, \cdots, x_m$ and decreasing with respect to x_{m+1}, $x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain set with a membership function μ where

$$\mu_l^{-1}(\alpha) = f\left(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)\right),$$
$$\mu_r^{-1}(\alpha) = f\left(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)\right).$$

One of the most important applications of uncertain set is linguistic summarizer, which is essentially a special uncertain proposition whose uncertain quantifier, uncertain subject and uncertain predicate are linguistic terms represented by uncertain sets. It is a human language statement that is concise and easy-to-understand by humans.

Definition 6. *(Liu [24]) Let the uncertain sets* $\mathcal{Q}, \mathcal{S}$ *and* $\mathcal{P}$ *denote the uncertain quantifier, uncertain subject, and uncertain predicate, respectively. Then the triplet*

$$(\mathcal{Q}, \mathcal{S}, \mathcal{P}) = \text{“}\mathcal{Q} \text{ of } \mathcal{S} \text{ are } \mathcal{P}''$$

is called an uncertain proposition.

For example, “Most young students are tall” is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is “most”, the uncertain subject $\mathcal{S}$ is “young students” and the uncertain predicate $\mathcal{P}$ is “tall”.

Theorem 2. *(Liu [24]) Let* $(\mathcal{Q}, \mathcal{S}, \mathcal{P})$ *be an uncertain proposition in which* $\mathcal{Q}$ *is a unimodal uncertain quantifier with membership function* λ*,* $\mathcal{S}$ *is an uncertain subject with membership function* ν*, and* $\mathcal{P}$ *is an uncertain predicate with membership function* μ*. Then the truth value of* $(\mathcal{Q}, \mathcal{S}, \mathcal{P})$ *with respect to the universe* A *is*

$$T(\mathcal{Q}, S, P) = \sup_{0 \le \omega \le 1} \left(\omega \wedge \sup_{K \in \mathbb{K}_\omega} \inf_{a \in K} \mu(a) \wedge \sup_{K \in \mathbb{K}_\omega^*} \inf_{a \in K} \neg\mu(a) \right)$$

where $\mathbb{K}_\omega = \{K \subset S_\omega \,|\, \lambda(|K|) \ge \omega\}\,, \mathbb{K}_\omega^* = \{K \subset S_\omega \,|\, \lambda(|S_\omega| - |K|) \ge \omega\}\,, S_\omega = \{a \in A \,|\, \nu(a) \ge \omega\}\,.$

3 Entropy of Uncertain Set

This section will introduce the definition of entropy for the membership function of uncertain set, and calculate the entropy of three different types of uncertain sets.

Definition 7. *(Liu [24]) Let* ξ *be an uncertain set with a membership function* μ*. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x$$

where $S(t) = -t \ln t - (1-t)\ln(1-t)$*.*

Liu [24] proved that the entropy of an uncertain set has translation invariance property, i.e.,

$$H[\xi + c] = H[\xi]$$

for any real number c.

Example 1. Let ξ be a rectangular uncertain set (a, b) with a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from the definition of entropy that

$$H[\xi] = \int_{-\infty}^{a} S(0)\mathrm{d}x + \int_{a}^{b} S(1)\mathrm{d}x + \int_{b}^{+\infty} S(0)\mathrm{d}x = 0.$$

Example 2. Let ξ be a triangular uncertain set (a,b,c) with a membership function

$$\mu(x)=\begin{cases}\dfrac{x-a}{b-a}, & \text{if } a\le x\le b\\ \dfrac{x-c}{b-c}, & \text{if } b\le x\le c\\ 0, & \text{otherwise.}\end{cases}$$

Then it follows from the definition of entropy that

$$\begin{aligned}H[\xi]&=\int_{-\infty}^{a}S(0)\mathrm{d}x+\int_{a}^{b}S\left(\frac{x-a}{b-a}\right)\mathrm{d}x+\int_{b}^{c}S\left(\frac{x-c}{b-c}\right)\mathrm{d}x+\int_{c}^{+\infty}S(0)\mathrm{d}x\\&=0+\frac{b-a}{2}+\frac{c-b}{2}+0\\&=\frac{c-a}{2}.\end{aligned}$$

Example 3. Let ξ be a trapezoidal uncertain set (a,b,c,d) with a membership function

$$\mu(x)=\begin{cases}\dfrac{x-a}{b-a}, & \text{if } a\le x\le b\\ 1, & \text{if } b\le x\le c\\ \dfrac{x-d}{c-d}, & \text{if } c\le x\le d\\ 0, & \text{otherwise.}\end{cases}$$

Then it follows from the definition of entropy that

$$\begin{aligned}H[\xi]&=\int_{-\infty}^{a}S(0)\mathrm{d}x+\int_{a}^{b}S\left(\frac{x-a}{b-a}\right)\mathrm{d}x+\int_{b}^{c}S(1)\mathrm{d}x+\int_{c}^{d}S\left(\frac{x-d}{c-d}\right)\mathrm{d}x+\int_{d}^{+\infty}S(0)\mathrm{d}x\\&=0+\frac{b-a}{2}+0+\frac{d-c}{2}+0\\&=\frac{b+d-a-c}{2}.\end{aligned}$$

4 Entropy Formula

Inverse membership function plays an important role in the operation of uncertain sets. In this section, we will give a formula to calculate the entropy of an uncertain set via its inverse membership function.

Theorem 3. *Assume ξ is an uncertain set with a regular membership function μ. If $H[\xi]$ exists, then*

$$H[\xi]=\int_0^1\left(\mu_l^{-1}(\alpha)-\mu_r^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Proof: Since ξ has a regular membership function μ, there exists a point x_0 such that $\mu(x_0)=1$. Then we have

$$\begin{aligned}H[\xi]&=\int_{-\infty}^{+\infty}S(\mu(x))\mathrm{d}x=\int_{-\infty}^{x_0}S(\mu(x))\mathrm{d}x+\int_{x_0}^{+\infty}S(\mu(x))\mathrm{d}x\\&=\int_{-\infty}^{x_0}\int_0^{\mu(x)}S'(\alpha)\mathrm{d}\alpha\mathrm{d}x+\int_{x_0}^{+\infty}\int_0^{\mu(x)}S'(\alpha)\mathrm{d}\alpha\mathrm{d}x\end{aligned}$$

where

$$S'(\alpha) = (-\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha))' = -\ln \frac{\alpha}{1-\alpha}.$$

It follows from Fubini theorem that

$$\begin{aligned}
H[\xi] &= \int_0^1 \int_{\mu_l^{-1}(\alpha)}^{x_0} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha + \int_0^1 \int_{x_0}^{\mu_r^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha \\
&= \int_0^1 \left(x_0 - \mu_l^{-1}(\alpha)\right) S'(\alpha)\mathrm{d}\alpha + \int_0^1 \left(\mu_r^{-1}(\alpha) - x_0\right) S'(\alpha)\mathrm{d}\alpha \\
&= \int_0^1 \left(\mu_r^{-1}(\alpha) - \mu_l^{-1}(\alpha)\right) S'(\alpha)\mathrm{d}\alpha \\
&= \int_0^1 \left(\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha)\right) \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha.
\end{aligned}$$

The theorem is thus verified.

Example 4. The triangular uncertain set $\xi = (a, b, c)$ has left and right inverse membership functions

$$\begin{aligned}
\mu_l^{-1}(\alpha) &= (b-a)\alpha + a, \\
\mu_r^{-1}(\alpha) &= (b-c)\alpha + c.
\end{aligned}$$

It follows from Theorem 3 that

$$\begin{aligned}
H[\xi] &= \int_0^1 ((b-a)\alpha + a - (b-c)\alpha - c) \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha \\
&= (c-a)\int_0^1 \alpha \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha + (a-c)\int_0^1 \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha \\
&= \frac{c-a}{2}.
\end{aligned}$$

Example 5. The trapezoidal uncertain set $\xi = (a, b, c, d)$ has left and right inverse membership functions

$$\begin{aligned}
\mu_l^{-1}(\alpha) &= (b-a)\alpha + a, \\
\mu_r^{-1}(\alpha) &= (c-d)\alpha + d.
\end{aligned}$$

It follows from Theorem 3 that

$$\begin{aligned}
H[\xi] &= \int_0^1 ((b-a)\alpha + a - (c-d)\alpha - d) \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha \\
&= (b+d-a-c)\int_0^1 \alpha \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha + (a-d)\int_0^1 \ln \frac{\alpha}{1-\alpha}\mathrm{d}\alpha \\
&= \frac{b+d-a-c}{2}.
\end{aligned}$$

The next theorem aims at providing a formula to calculate the entropy of a function of independent uncertain sets. It will also be used to verify the linearity of the entropy operator for independent uncertain sets in the next section.

Theorem 4. *Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain sets with regular membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to x_1, x_2, $\cdots, x_m$, and strictly decreasing with respect to $x_{m+1}, \cdots, x_n$, then*

$$H[f(\xi_1, \xi_2, \cdots, \xi_n)] = \int_0^1 \left(f\left(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)\right) \right.$$
$$\left. - f\left(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)\right)\right) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Proof: Note that $f(\xi_1, \xi_2, \cdots, \xi_n)$ has a membership function μ where

$$\mu_l^{-1}(\alpha) = f\left(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)\right),$$
$$\mu_r^{-1}(\alpha) = f\left(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)\right)$$

by Theorem 1. Since

$$H[f(\xi_1, \xi_2, \cdots, \xi_n)] = \int_0^1 \left(\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha)\right) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha,$$

by Theorem 3, the theorem follows immediately.

Example 6. Let $\xi_1 = (a_1, b_1, c_1)$ and $\xi_2 = (a_2, b_2, c_2)$ be two independent triangular uncertain sets with $a_1 \geq 0$ and $a_2 \geq 0$. Consider the entropy of the uncertain set $\xi = \xi_1\xi_2$. Note that ξ_i has left and right inverse membership functions

$$\mu_{il}^{-1}(\alpha) = (b_i - a_i)\alpha + a_i,$$
$$\mu_{ir}^{-1}(\alpha) = (b_1 - c_1)\alpha + c_i$$

for $i = 1, 2$. Then the uncertain set ξ has left and right inverse membership functions

$$\mu_l^{-1}(\alpha) = (b_1 - a_1)(b_2 - a_2)\alpha^2 + (a_1b_2 + a_2b_1 - 2a_1a_2)\alpha + a_1a_2,$$
$$\mu_r^{-1}(\alpha) = (b_1 - c_1)(b_2 - c_2)\alpha^2 + (c_1b_2 + c_2b_1 - 2c_1c_2)\alpha + c_1c_2.$$

It follows from Theorem 4 that

$$H[\xi] = \int_0^1 \left(\mu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha)\right) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = \frac{c_1c_2 - a_1a_2}{2}.$$

5 Positive Linearity of Entropy Operator

In this section, we will verify an important property of the entropy operator for uncertain sets, that is positive linearity property.

Theorem 5. *(Positive Linearity) Let ξ and η be independent uncertain sets with regular membership functions. Then for any real number a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta].$$

Proof: Suppose ξ and η have regular membership functions μ and ν, respectively. The theorem will be proved in three steps.

Step 1: We prove $H[a\xi]=|a|H[\xi]$. If $a\ge 0$, then the left and right inverse membership functions of $a\xi$ are

$$\lambda_l^{-1}(\alpha)=a\mu_l^{-1}(\alpha),\quad \lambda_r^{-1}(\alpha)=a\mu_r^{-1}(\alpha).$$

It follows from Theorem 4 that

$$H[a\xi]=\int_0^1\left(a\mu_l^{-1}(\alpha)-a\mu_r^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=a\int_0^1\left(\mu_l^{-1}(\alpha)-\mu_r^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=|a|H[\xi].$$

If $a=0$, we have $H[a\xi]=0=|a|H[\xi]$. If $a<0$, then the left and right inverse membership functions of $a\xi$ are

$$\lambda_l^{-1}(\alpha)=a\mu_r^{-1}(\alpha),\quad \lambda_r^{-1}(\alpha)=a\mu_l^{-1}(\alpha).$$

It follows from Theorem 4 that

$$H[a\xi]=\int_0^1\left(a\mu_r^{-1}(\alpha)-a\mu_l^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=(-a)\int_0^1\left(\mu_l^{-1}(\alpha)-\mu_r^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha=|a|H[\xi].$$

Thus we have $H[a\xi]=|a|H[\xi]$.

Step 2: We prove $H[\xi+\eta]=H[\xi]+H[\eta]$. Note that the left and right inverse membership functions of $\xi+\eta$ are

$$\lambda_l^{-1}(\alpha)=\mu_l^{-1}(\alpha)+\nu_l^{-1}(\alpha),\quad \lambda_r^{-1}(\alpha)=\mu_r^{-1}(\alpha)+\nu_r^{-1}(\alpha).$$

It follows from Theorem 4 that

$$\begin{aligned}H[\xi+\eta]&=\int_0^1\left(\mu_l^{-1}(\alpha)+\nu_l^{-1}(\alpha)-\mu_l^{-1}(\alpha)-\nu_r^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha\\&=\int_0^1\left(\mu_l^{-1}(\alpha)-\mu_l^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha+\int_0^1\left(\nu_l^{-1}(\alpha)-\nu_l^{-1}(\alpha)\right)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha\\&=H[\xi]+H[\eta].\end{aligned}$$

Step 3: For any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi+b\eta]=H[a\xi]+H[b\eta]=|a|H[\xi]+|b|H[\eta].$$

The theorem is proved.

Example 7. Let $\xi_1=(a_1,b_1,c_1)$ and $\xi_2=(a_2,b_2,c_2)$ be two independent triangular uncertain sets. Consider the entropy of the uncertain set $\xi=\xi_1+\xi_2$. Note that

$$H[\xi_i]=\frac{c_i-a_i}{2}$$

for $i=1,2$. It follows from Theorem 5 that

$$H[\xi]=H[\xi_1]+H[\xi_2]=\frac{c_1-a_1}{2}+\frac{c_2-a_2}{2}=\frac{c_1+c_2-a_1-a_2}{2}.$$

6 Relative Entropy Operator

Relative entropy was first proposed by Kullback and Leibler [17] in 1951 to measure the divergence between two probability distributions. After that, Bhandari and Pal [3] introduced relative entropy to fuzzy sets in 1993. Then Chen *et al.* [5] introduced relative entropy to uncertain variables. In this section, we will propose a concept of relative entropy for the membership function of an uncertain set.

Definition 8. *Let ξ and η be two uncertain sets with membership functions μ and ν, respectively. Then the relative entropy of ξ and η is defined by*

$$R[\xi,\eta]=\int_0^1 \mu(x)\ln\left(\frac{\mu(x)}{\nu(x)}\right)+\nu(x)\ln\left(\frac{\nu(x)}{\mu(x)}\right)\mathrm{d}x.$$

Remark 1. The relative entropy operator is symmetric, i.e., $R[\xi,\eta]=R[\eta,\xi]$ for any two uncertain sets ξ and η.

Theorem 6. *Let ξ and η be two uncertain sets. Then $R[\xi,\eta]\geq 0$ and the equality holds if and only if ξ and η have the same membership function.*

Proof: Define a binary function

$$r(s,t)=s\ln\left(\frac{s}{t}\right)+t\ln\left(\frac{t}{s}\right)=(s-t)(\ln s-\ln t)$$

on $[0,1]\times[0,1]$. Assume the uncertain sets ξ and η have membership functions μ and ν, respectively. Then

$$R[\xi,\eta]=\int_0^1 r(\mu(x),\nu(x))\,\mathrm{d}x.$$

Since $\ln x$ is a monotone increasing function with respect to x, we have $r(s,t)\geq 0$ for any $(s,t)\in[0,1]\times[0,1]$. As a result, $R[\xi,\eta]\geq 0$. Besides, $r(s,t)=0$ if and only if $s=t$, so $R[\xi,\eta]=0$ if and only if ξ and η have the same membership function.

Example 8. Let $\xi_1=(a,b_1,c)$ and $\xi_2=(a,b_2,c)$ be two triangular uncertain sets with $b_1<b_2$. Since ξ_i has a membership function

$$\mu_i(x)=\begin{cases}(x-a)/(b_i-a), & \text{if } a\leq x\leq b_i\\ (x-c)/(b_i-c), & \text{if } b_i\leq x\leq c\end{cases}$$

for $i=1,2$, we have

$$\begin{aligned}R[\xi_1,\xi_2]=&\int_a^{b_1}\frac{x-a}{b_1-a}\ln\left(\frac{x-a}{b_1-a}\Big/\frac{x-a}{b_2-a}\right)+\frac{x-a}{b_2-a}\ln\left(\frac{x-a}{b_2-a}\Big/\frac{x-a}{b_1-a}\right)\mathrm{d}x\\&+\int_{b_1}^{b_2}\frac{x-c}{b_1-c}\ln\left(\frac{x-c}{b_1-c}\Big/\frac{x-a}{b_2-a}\right)+\frac{x-a}{b_2-a}\ln\left(\frac{x-a}{b_2-a}\Big/\frac{x-c}{b_1-c}\right)\mathrm{d}x\\&+\int_{b_2}^{c}\frac{x-c}{b_1-c}\ln\left(\frac{x-c}{b_1-c}\Big/\frac{x-c}{b_2-c}\right)+\frac{x-c}{b_2-c}\ln\left(\frac{x-c}{b_2-c}\Big/\frac{x-c}{b_1-c}\right)\mathrm{d}x.\end{aligned}$$

Specially, we assume $b_1=2/3a+1/3c$ and $b_2=1/3a+2/3c$, then

$$R[\xi_1,\xi_2]=\frac{c-a}{2}-\frac{c-a}{2}\ln 2.$$

7 Conclusions

This paper mainly studied the entropy operator of an uncertain set. Firstly, we proposed a formula in the form of inverse membership functions to calculate the entropy of an uncertain set. By this formula, we verified the positive linearity property of entropy for uncertain sets. In addition, a concept of relative entropy was introduced to measure the divergence between the membership functions of two uncertain sets. Further research may include the applications of entropy to clustering, pattern recognition, image compression, and some other areas.

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