Hamilton Index and Its Algorithm of Uncertain Graph

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Abstract: In practical applications of graph theory, uncertain factors are frequently encountered. In this paper, the uncertainty theory is employed to deal with uncertain factors in uncertain graph, in which whether two vertices are joined by an edge cannot be completely determined. In order to show how likely an uncertain graph is Hamiltonian, the concept of Hamilton index of uncertain graph is proposed, and then some properties of the Hamilton index are discussed. Besides, we present an algorithm to calculate the Hamilton index of uncertain graph.

Keywords: Uncertainty theory; Uncertain measure; Uncertain variable; Hamilton index; Uncertain graph

1 Introduction

Many real world situations can conveniently be described by means of a graph consisting of a set of vertices together with edges joining certain pairs of these vertices. For example, the vertices could represent people, with edges joining pairs of friends; or the vertices might be intersections, with edges representing the streets connecting the two junctions.

The study of graph theory can be traced back to the eighteenth century. Euler written a paper on the seven bridges of Königsberg and published in 1736. This paper is regarded as the first paper in the history of graph theory. Since then, many researchers, such as Bermond and Thomassen [2], Bondy and Murty [4], Chiue and Shieh [6], Harary [13], Tutte [24] have done much work in the field of graph theory.

In classic graph theory, the edges and the vertices are all deterministic. However, in practical application, as the system becomes more complex, indeterminacy factors may appear in graphs, which leads to new situations. Such as in minimum spanning tree problem, if the weight of each edge is indeterminacy, the optimization objective can not be dealt with by classical theory and conventional algorithms directly.

Sometimes, whether two vertices are joined by an edge cannot be completely determined in a graph. Then, how does one deal with these indeterminacy factors? Some researchers introduced probability theory into the graph theory. Random graphs were first defined by Erdös and Rényi [7, 8]. They thought

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that whether two vertices are joined can be described as a random variable. Under the assumption, Gilbert [12] studied the probability that the random graph is connected, and also the probability that two specific vertices are connected. Luczak [21] studied the behavior of a random graph process. Further research on random graph, we may consult Barabási and Albert [1], Bollobás [3], Mahmoud et al. [22], etc.

However, it is not suitable to regard every indeterminacy phenomenon as random phenomenon. A fundamental premise of applying probability theory is that the sample size is large enough, and the estimated probability is close enough to the real frequency. Otherwise, the probability theory is no longer applicable. Unfortunately, very often we are frequently lack of observed data about the unknown state of nature, due to economic reasons or technical difficulties. Usually, we may ask some domain experts to give a belief degree of the possible events. Since human beings usually overweight unlikely events (Kahneman and Tversky [14]), the belief degree usually has much larger range than the true frequency, so the probability theory is no longer valid. In order to deal with this uncertain phenomenon, Liu [16] proposed the uncertainty theory in 2007 and refined by Liu [19] in 2010, which has become a branch of axiomatic mathematics.

In theoretical aspect, Gao [9] gave some mathematical properties of continuous uncertain measure. Li and Liu [15] proposed the concept of uncertain logic. Chen and Liu [5] proved the existence and uniqueness theorem for uncertain differential equations. In addition, Liu and Ha [20] gave the important formula of the expected value of function of uncertain variables. Peng and Yao [23] proposed a mean reversion uncertain stock model and also given the corresponding option pricing formulas.

From a practical point of view, uncertain programming was founded by Liu [18]. Yan [25] provided two models for portfolio selection in which the securities are assumed to be uncertain variables. Gao [11] investigated the shortest path problem with uncertain arc lengths. Zhang and Peng [26] presented some uncertain programming models for Chinese postman problem in uncertain environment.

As an efficient tool of modeling the behavior of uncertain phenomena, uncertainty theory is employed to deal with uncertain in graph. In 2011, Gao and Gao [10] proposed the concept of uncertain graph, and investigated the connectedness index of uncertain graph. Recently, Zhang and Peng [27] given an Euler index to show how likely an uncertain graph is Eulerian.

In an uncertain graph, whether two vertices of the graph are joined by an edge cannot be completely determined. Then, to an uncertain graph, at how much belief degree we can regard the graph is Hamiltonian? In this paper, the concept of Hamilton index of uncertain graph is firstly proposed. Under the framework of uncertainty theory, an algorithm to calculate Hamilton index of uncertain graph is given.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertainty theory and graph theory are introduced. In Section 3, the concept of Hamilton index of uncertain graph is proposed; After that, some properties of the Hamilton index are discussed. In Section 4, an algorithm to calculate the Hamilton index of uncertain graph is given, and then an example is given to illustrate the algorithm. Section 5 concludes this paper with a brief summary.

2 Preliminaries

2.1 Uncertainty Theory

In this section, we present some basic concepts and results from uncertainty theory, which will be used throughout this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. For any $\Lambda \in \mathcal{L}$, $\mathcal{M}\{\Lambda\}$ is called an uncertain measure if it satisfies the following three axioms

(1)(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;

(2)(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

(3)(Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

In order to obtain an uncertain measure of compound event, the fourth axiom called product axiom was defined by Liu [17] as follows

(4)(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

A function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. An uncertain variable is said to be Boolean if it takes values either 0 or 1.

A Boolean function $f(B_1, B_2, \cdots, \{0\}, \cdots, B_n)$ is said to be an increasing Boolean function if

$$f(B_1, B_2, \cdots, \{0\}, \cdots, B_n) = 1 \Longrightarrow f(B_1, B_2, \cdots, \{0,1\}, \cdots, B_n) = 1.$$

Theorem 1 *(Liu [19]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1, & \text{with uncertain measure } \alpha_i \\ 0, & \text{with uncertain measure } 1 - \alpha_i \end{cases}$$

for $i = 1, 2, \cdots, n$. If f is a Boolean function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\leq i\leq n}\nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\leq i\leq n}\nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0}\min\limits_{1\leq i\leq n}\nu_i(x_i), & \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1}\min\limits_{1\leq i\leq n}\nu_i(x_i) \geq 0.5 \end{cases}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} \alpha_i, & \text{if } x_i = 1 \\ 1 - \alpha_i, & \text{if } x_i = 0 \end{cases}$$

for $i = 1, 2, \cdots, n$, respectively.

In more detail, if f is an increasing Boolean function, we have the following theorem.

Theorem 2 *(Gao and Gao [10]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1, & \text{with uncertain measure } \alpha_i \\ 0, & \text{with uncertain measure } 1 - \alpha_i \end{cases}$$

for $i = 1, 2, \cdots, n$. If f is an increasing Boolean function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(B_1, B_2, \cdots, B_n) = 1} \min_{1 \le i \le n} \mathcal{M}\{\xi_i \in B_i\},$$

where B_i are subsets of $\{0, 1\}$, $i = 1, 2, \cdots, n$.

2.2 Hamilton Graphs

Now, we introduce some basic concepts of graph theory. The following concepts and terminology in this subsection come from Bondy and Murty [4].

Definition 1 *(Bondy and Murty [4]) A graph G is an order triple $(V(G), E(G), \psi_G)$ consisting of a nonempty vertex set $V(G)$, a set $E(G)$ of edges, and an incidence function ψ_G that associates with each edge an unordered pair of (not necessarily distinct) vertices of G.*

Let G be a graph, a loop is an edge with identical ends, and a link is an edge with distinct ends. A simple graph is a graph having no loops and no two of its links join the same pair of vertices. Graph G is finite if both its vertex set and edge set are finite. In this paper, the graphs are all simple finite graphs.

For any given graph G there corresponds an adjacency matrix of G. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. Then the adjacency matrix of G is the $n \times n$ matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

where

$$x_{ij} = \begin{cases} 1, & \text{if there exists an edge between vertices } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Note that $x_{ii} = 0$ and $x_{ij} = x_{ji}$ for $i, j = 1, 2, \cdots, n$, respectively, or say, X is a symmetric matrix.

Let G be a graph, an uv-walk of length k in G is a sequence $v_0 e_1 v_1 e_2 \cdots e_k v_k$, where $v_0 = u$, $v_k = v$, whose terms are alternately vertices and edges, the edge e_i has ends v_{i-1} and v_i. An uv-trail in G is an uv-walk with no repeated edge. An uv-path in G is an uv-walk with no repeated vertex.

A walk is closed if it has positive length and its origin and terminus are the same. A cycle is a closed trail whose origin and internal vertices are distinct.

Let G be a graph, a Hamilton path is a path that contains all vertices of the graph, and a Hamilton cycle of G is a cycle that contains every vertex of G. A graph is Hamiltonian if it contains a Hamilton cycle.

3 Hamilton Index

3.1 Concepts

In classic graph theory, the edges and vertices are all deterministic, either existing or not. However, as the system becomes more complex, some indeterminacy factors will appear in graphs. Sometimes, whether two vertices are joined by an edge can not be completely determined. If there is enough historical data of the information about it to estimate the distribution via statistics, then we can regard the event that whether two vertices are joined by an edge as a random event.

However, due to economic reasons or technical difficulties, very often we are lack of history data, or history data is invalid because of unexpected events have occurred. In this case, we may ask some domain experts to give a belief degree that the edge exists. As we stated before, this expert data is just the subject of uncertainty theory.

In 2011, Gao and Gao [10] proposed the concept of uncertain graph in which all edges are independent and exist with some belief degrees in uncertain measure.

Definition 2 *(Gao and Gao [10]) A graph of order n is said to be uncertain if it has uncertain adjacency matrix*

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

where α_{ij} represent that the edges between vertices v_i and v_j exist with uncertain measures α_{ij}, $i, j = 1, 2, \cdots, n$, respectively.

Note that $\alpha_{ii} = 0$ and $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \cdots, n$, respectively, which means A is a symmetric matrix.

The number of vertices in G is called the order of G, while the number of uncertain edges is called its size. The elements of uncertain adjacency matrix are no longer 0 or 1 but numbers in $[0, 1]$. For example, $\alpha_{ij} = 0.8$ indicates the edge between vertices v_i and v_j exists with uncertain measure 0.8 and dose not exist with uncertain measure 0.2.

According to the definition of uncertain graph, we have that the edge set of uncertain graph G is a set of Boolean uncertain variable $E(G) = \{\xi_{12}, \xi_{13}, \cdots, \xi_{1n}, \xi_{23}, \cdots, \xi_{2n}, \cdots, \xi_{(n-1)n}\}$, where $\mathcal{M}\{\xi_{ij} = 1\} = \alpha_{ij}$ and $\mathcal{M}\{\xi_{ij} = 0\} = 1 - \alpha_{ij}$, for $1 \leq i < j \leq n$. Generally speaking, remove the edges ξ_{ij} satisfying $\mathcal{M}\{\xi_{ij} = 1\} = 0$, and denote $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$.

Notice that in such graphs one is mainly interested in whether or not two given vertices are joined by an edge.

Definition 3 *Let G be an uncertain graph with edge set $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. The underlying graph of G, denoted by $\underline{G}$, is a graph obtained from G by replacing each edge by $\mathcal{M}\{\xi_{ij} = 1\} = 1$.*

In the following, we give the concept of Hamilton function.

Definition 4 *Assume that G is an uncertain graph with edge set $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. The Hamilton function of G is denoted as:*

$$H(E(G)) = \begin{cases} 1, & \text{if graph } G \text{ is Hamiltonian} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $H(E(G))$ is an increasing Boolean function. In order to show how likely an uncertain graph is Hamiltonian, Hamilton index is defined below.

Definition 5 *The Hamilton index $\rho(G)$ of an uncertain graph G is the uncertain measure that the uncertain graph is Hamiltonian, i.e.,*

$$\mathcal{M}\{H(E(G)) = 1\}.$$

Remark 1 The value of $\rho(G)$ represents the belief degree that the uncertain graph G is Hamiltonian. $\rho(G) = 1$ means that the graph is completely Hamiltonian, and $\rho(G) = 0$ means that the graph is absolutely non-Hamiltonian. Thus the larger the value of $\rho(G)$ is, the more true G is Hamiltonian.

Remark 2 If an uncertain graph G is Hamiltonian with belief degree α, then the graph G is non-Hamiltonian with belief degree $1 - \alpha$. This fact follows from the duality property of uncertain measure. In other words, if $\rho(G) = \alpha$, then

$$\mathcal{M}\{H(E(G)) \neq 1\} = 1 - \mathcal{M}\{H(E(G)) = 1\} = 1 - \alpha.$$

Remark 3 If the underlying graph $\underline{G}$ of the uncertain graph G is non-Hamiltonian, then the Hamilton index of G is $\rho(G) = 0$.

3.2 Properties

How can we calculate the Hamilton index when an uncertain graph is given? The following theorem completely solves this problem.

Theorem 3 *Let G be an uncertain graph of order n and its uncertain adjacency matrix is*

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix},$$

where $\alpha_{ii} = 0$ *and* $\alpha_{ij} = \alpha_{ji}$ *for* $i, j = 1, 2, \cdots, n$, *respectively. If all edges are independent, then the Hamilton index of* G *is*

$$\rho(G) = \begin{cases} \sup\limits_{H(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{H(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{H(X)=0} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{H(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

$x_{ij} \in \{0, 1\}$, $x_{ij} = x_{ji}$, $x_{ii} = 0$, *and* ν_{ij} *are defined by*

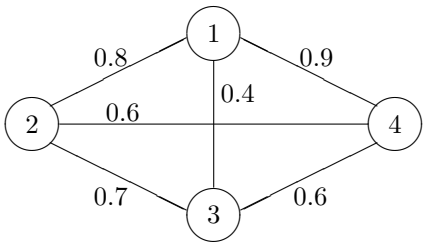
$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases}$$

for $i, j = 1, 2, \cdots, n$, *respectively, and*

$$H(X) = \begin{cases} 1, & \text{if } X \text{ is Hamiltonian} \\ 0, & \text{otherwise.} \end{cases}$$

Proof It is clear that G has a Hamilton cycle if and only if $H(X) = 1$. Note that $H(X)$ is a Boolean function. It follows from Theorem 1 that the theorem is proved.

Example 1 Figure 1 illustrates the uncertain graph G of order 4 size 6 and its uncertain adjacency matrix.



$$\begin{pmatrix} 0 & 0.8 & 0.4 & 0.9 \\ 0.8 & 0 & 0.7 & 0.6 \\ 0.4 & 0.7 & 0 & 0.6 \\ 0.9 & 0.6 & 0.6 & 0 \end{pmatrix}$$

Figure 1: Uncertain graph G and its uncertain adjacency matrix

Since the size of the uncertain graph is 6 in Figure 1, its adjacency matrix breaks down into $2^6 = 64$ cases.

Assume the adjacency matrix X of G is one of the following ten matrices

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then we have the graph G is Hamiltonian, i.e., $H(X) = 1$ and

$$\sup_{H(X)=1} \min_{1\le i<j\le 4} \nu_{ij}(X) = 0.6.$$

And assume X is one of the remaining fifty-four matrices, then we have the graph G is non-Hamiltonian, i.e., $H(X) = 0$ and

$$\sup_{H(X)=0} \min_{1\le i<j\le 4} \nu_{ij}(X) = 0.4.$$

It follows from Theorem 3 that the Hamilton index of G

$$\rho(G) = 1 - \sup_{H(X)=0} \min_{1\le i<j\le 4} \nu_{ij}(X) = 0.6.$$

In more detail, since $H(E(G))$ is an increasing Boolean function, Theorem 2 directly leads to

Theorem 4 *Let G be an uncertain graph with edge set $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. The Hamilton index of G is*

$$\rho(G) = \sup_{H(E(G))=1} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

where B_i are subsets of $\{0, 1\}$, $i = 1, 2, \cdots, m$.

Obviously, according to Theorem 4, we have

Corollary 1 *Let G be an uncertain cycle with edge set $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. Then the Hamilton index of G is the smallest value of $\mathcal{M}\{\xi_i = 1\}$, $i = 1, 2, \cdots, m$.*

For an uncertain graph G, we can construct an uncertain complete graph K_n by add the edges ξ_{ij} satisfying $\mathcal{M}\{\xi_{ij} = 1\} = 0$ to G such that the underlying graph $\underline{K_n}$ of K_n is a complete graph (i.e., each pair of distinct vertices is joined by an edge).

Remark 4 Assume G is an uncertain graph, K_n is an uncertain complete graph obtained from G. It is clear that the Hamilton index of the uncertain graph G is equal to the Hamilton index of the uncertain complete graph K_n, i.e.,

$$\rho(G) = \rho(K_n).$$

Definition 6 *Assume that an uncertain complete graph K_n has vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(K_n) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. The Hamilton cycle of K_n is an uncertain cycle with vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(H) = \{\xi_{h_1}, \xi_{h_2}, \cdots, \xi_{h_n}\}$. The maximum index Hamilton cycle is a Hamilton cycle with the maximum Hamilton index.*

In the following, we will show that the Hamilton index of an uncertain complete graph K_n is equal to the Hamilton index of the maximum index Hamilton cycle of K_n.

Theorem 5 *Let K_n be an uncertain complete graph with vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(K_n) = \{\xi_1, \xi_2, \cdots, \xi_m\}$, and H^* be the maximum index Hamilton cycle with vertex set $V(H^*) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(H^*) = \{\xi_{h_1}, \xi_{h_2}, \cdots, \xi_{h_n}\}$. Then we have*

$$\rho(K_n) = \rho(H^*).$$

Proof Clearly, $\rho(K_n) \geq \rho(H^*)$. Next, we will prove $\rho(K_n) \leq \rho(H^*)$, where $\rho(K_n) > 0$.

Since $H(E(K_n))$ is an increasing Boolean function, there must exist a series of $\{B_i'\}_{i=1}^m$, taking values of $\{1\}$ or $\{0, 1\}$, satisfying

$$\rho(K_n) = \sup_{H(E(K_n))=1} \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i\} = \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i'\} = \min_{1\leq k\leq t} \mathcal{M}\{\xi_{i_k} = 1\} > 0, \tag{1}$$

where t is a positive number and subseries $\{B_{i_k}'\}_{k=1}^t$ takes values of $\{1\}$.

Obviously, $t \geq n$. In fact, we can choose $\{B_{i_k}'\}_{k=1}^t$ such that $t = n$. If $t = n$, we have done.

If $t > n$, the subgraph S with vertex set $V(S) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(S) = \{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_t}\}$ must contain at least a Hamilton cycle and some other edges, the set of these edges is denoted by A. Choose any edge of A, such as ξ_{i_t}, and remove it. Then we obtain a new subgraph S_1 with vertex set $V(S_1) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(S_1) = \{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_{t-1}}\}$. Furthermore, S_1 must contain a Hamilton cycle and

$$\rho(K_n) \geq \rho(S_1) \geq \min_{1\leq k\leq t-1} \mathcal{M}\{\xi_{i_k} = 1\} \geq \min_{1\leq k\leq t} \mathcal{M}\{\xi_{i_k} = 1\}. \tag{2}$$

Combining with (1) and (2), we have

$$\rho(K_n) = \rho(S_1).$$

Repeating this argument until there is only a Hamilton cycle. That is, we obtain a Hamilton cycle H such that

$$\rho(K_n) = \rho(H).$$

Since H^* is the maximum index Hamilton cycle, thus

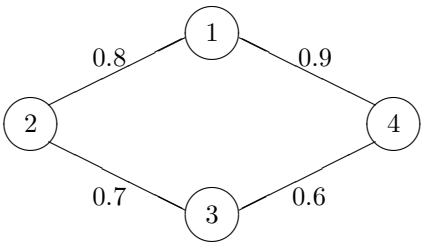
$$\rho(K_n) = \rho(H) \leq \rho(H^*).$$

The theorem is proved.

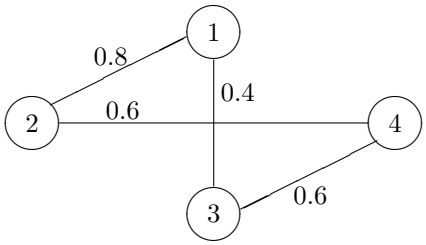
Remark 5 Assume G is an uncertain graph, K_n is an uncertain complete graph obtained from G. Theorems 5 tells us that in order to calculus the Hamilton index of G, we only need to find the maximum index Hamilton cycle of K_n.

Example 2 Assume that G is an uncertain graph presented in Figure 1. Obviously, G is also an uncertain complete graph. There are three Hamilton cycles in the uncertain graph G (see Figure 2).

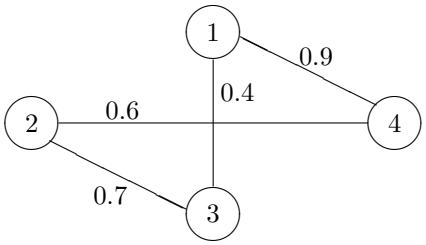
According to Corollary 1, we obtain the Hamilton index of the Hamilton cycles shown in Figure 2 as follows: $\rho(H_1) = 0.6$, $\rho(H_2) = 0.4$, $\rho(H_3) = 0.4$. From Theorem 5, the Hamilton index of the uncertain graph in Figure 1 is 0.6.



(2): Hamilton cycle H_1



(3): Hamilton cycle H_2



(4): Hamilton cycle H_3

Figure 2: Hamilton cycles for Example 2

4 Algorithm and Example

Obviously, finding the maximum index Hamilton cycle is extremely similar to the traveling salesman problem, which has been proved to be an NP-hard problem. For small-scale uncertain graph, an immediately obvious method is to enumerate all Hamilton cycles and then comparison to find the maximum index Hamilton cycle by Corollary 1. For large-scale uncertain graph, we will give an approximation algorithm to calculus the Hamilton index, or say, find the maximum index Hamilton cycle.

Assume that uncertain graph G has vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. We construct an uncertain complete graph K_n by add the edges ξ_{ij} satisfying $\mathcal{M}\{\xi_{ij} = 1\} = 0$ to G. There always exists a Hamilton cycle in K_n, one such is the cycle given by $\{v_1 v_2 \cdots v_n v_1\}$.

Algorithm

Step 1. Constructing an uncertain complete graph K_n and finding a Hamilton cycle $H^* = \{v_1 v_2 \cdots v_n v_1\}$ in K_n.

Step 2. For some i and j such that $1 < i + 1 < j < n$, if

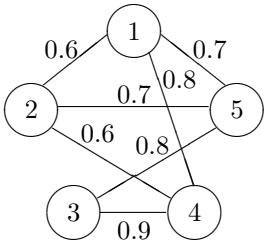
$$\min\{\mathcal{M}\{\xi_{ij} = 1\}, \mathcal{M}\{\xi_{(i+1)(j+1)} = 1\}\} > \min\{\mathcal{M}\{\xi_{i(i+1)} = 1\}, \mathcal{M}\{\xi_{j(j+1)} = 1\}\},$$

$H^* \leftarrow \{v_1 v_2 \cdots v_i v_j v_{j-1} \cdots v_{i+1} v_{j+1} v_{j+2} \cdots v_n v_1\}$.

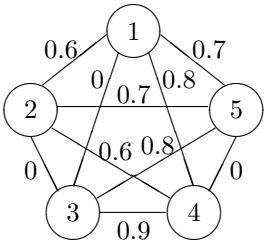
Step 3. Repeat the Step 2 until one is left with a cycle that can be improved no more by this method.

Step 4. Report the last H^* as the maximum index Hamilton cycle.

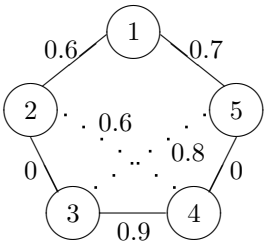
Figure 3 gives an example. According to Corollary 1, the Hamilton index of the uncertain graph in Figure 3(1) is 0.6. For more accuracy, the procedure can be repeated several times, starting with a different cycle each time.



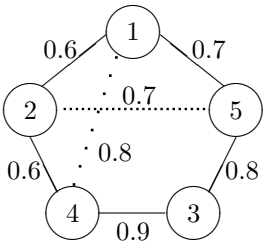
(1): Uncertain graph G



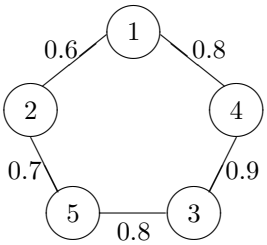
(2): Uncertain complete graph K_5



(3): Hamilton cycle H_1^*



(4): Hamilton cycle H_2^*



(5): Hamilton cycle H_3^*

Figure 3: Maximum index Hamilton cycle by the Algorithm

5 Conclusion

This paper employed uncertainty theory to deal with the Hamilton cycle in uncertain graph. In order to show how likely an uncertain graph is Hamiltonian, i.e., the graph contains a Hamilton cycle, the concept of Hamilton index was proposed, and some properties of the Hamilton index were investigated. What is more, an algorithm to calculate the Hamilton index was given.

It is worth pointing out that this paper only investigated how likely a graph is Hamiltonian under uncertain environment. The problem in other more complex environments, such as uncertain random environment, may become new topics in further research.

Acknowledgements

This work is supported by the National Natural Science Foundation (No.60874067), the Hubei Provincial Natural Science Foundation (No.2010CDB02801), and the Scientific and Technological Innovation Team Project (No.T201110) of Hubei Provincial Department of Education, China.

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