The Pricing of Vulnerable Options for Uncertain Financial Market

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Abstract

Uncertainty theory is a branch of axiomatic mathematics for modeling human uncertainty. Uncertain finance is developed based on uncertainty theory. In this paper, the vulnerable options pricing problems for uncertain financial market are investigated. Firstly, the corporate debt pricing model with default risk is built. Then, European vulnerable call and put option models are proposed. Some mathematical properties of these models are studied.

Keywords: project scheduling problem, uncertain programming, uncertainty theory, genetic algorithm

1 Introduction

In real life, human language like “about 100km”, “approximately 39°C”, “roughly 80kg”, “low speed”, “middle age”, and “big size” are usually used to express the imprecise information and knowledge. Perhaps in a long period of time these imprecise quantities are considered as subjective probability or fuzziness. However, a lot of surveys showed that they are neither like randomness nor like fuzziness. This fact provides a motivation to invent uncertainty theory founded by Liu [10] which is a branch of axiomatic mathematics for modeling human uncertainty based on normality, monotonicity, duality, subadditivity and product axioms. Since the superiority of the dealing human uncertainty (Liu [16]), uncertainty theory has been successfully applied to many fields. Uncertain programming, a type of mathematical programming involving uncertain variables, was proposed by Liu [13]. Besides, Li and Liu [9] proposed uncertain logic, which can be seen as a consistent generalization of multi-valued logics. In order to provide a methodology for collecting and interpreting expert’s experimental data by uncertainty theory, uncertain statistics was started by Liu [14] in 2011. For exploring the recent developments of uncertainty theory, the readers may consult Liu [14].

In the early 1970s, Black and Scholes [2] and, independently, Merton [18] constructed a theory for determining the stock option price which is the famous Black-Scholes formula. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. As a different doctrine, based on the assumption that stock price follows a geometric canonical process, uncertainty theory was first introduced into finance by Liu [12] in 2009. The core part canonical process (Liu [12]) which is a Lipschitz continuous uncertain process with stationary and independent increments can be seen as a counterpart of Brownian motion. In addition, Liu [11] gave the definition of uncertain differential equation, a type of differential equation driven by canonical process. After that, Chen and Liu [3] proved an existence and uniqueness of solution for uncertain differential equation. Furthermore, Liu [11] derived an uncertain stock model. Based on this model, European call and put option price formulae were proposed by Liu [12]. Since then, American call and put option pricing formulae were given by Chen [4]. Besides, Peng and Yao [20] proposed a different uncertain stock model and other option price formulae. Furthermore, uncertainty theory was introduced to insurance models by Liu [17] based on the assumption that the claim process is an uncertain renewal reward process.

However, all these uncertain option models have no default risk. For there are many options and financial assets sold by firms with limited assets. And in the real financial market, a large number of options are written privately and are not guaranteed by a third party. Some of the contracts, like insurance contract can be considered as a vulnerable put option. The studying vulnerable option is started by Merton [19] in which he investigated the risk structure of interest rate. Besides, Johson and Stulz [7] established vulnerable call and put option pricing models. The other researchers like Klein [8] and Hull and White [6] carried a further research on this model. In this paper, the vulnerable option pricing formulae are derived for uncertain financial market and some mathematical properties of them are discussed. The rest of the paper is organized as follows. Some preliminary concepts of uncertain process are recalled in Section 2. Corporate debt pricing model with default risk is built in Sections 3. The European vulnerable call and put option formulae are carried out and
some simple properties are discussed in Sections 4 and 5, respectively. Finally, a brief summary is given in Section 6.

2 Preliminary

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu [11] in 2008.

Definition 1 (Liu [11]) Let $T$ be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set $B$ of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$$

is an event.

Definition 2 (Liu [12]) An uncertain process $C_t$ is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3t}}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$ 

If $C_t$ is canonical process, then the uncertain process $X_t = \exp(et + \sigma C_t)$ is called a geometric canonical process.

An assumption that the stock price follows geometric canonical process was presented by Liu [12]. In Liu’s stock model, the bond price $X_t$ and the stock price $Y_t$ are determined by

$$\begin{align*}
\frac{dX_t}{X_t} &= rd dt \\
\frac{dY_t}{Y_t} &= eY_t dt + \sigma Y_t dC_t
\end{align*}$$

(1)

where $r$ is the riskless interest rate, $e$ is the stock drift, $\sigma$ is the stock diffusion, and $C_t$ is a canonical process.

Option pricing problem is a fundamental problem in financial market. European option is the most classic and useful option. European call and put option pricing formulae were proposed by Liu [12] for Liu’s stock model. Besides, American call and put option pricing formulae were proposed by Chen [4] for Liu’s stock model.

3 Corporate Debt Pricing with Default Risk

Suppose that the value of a firm is $V_t$ at time $t$ and it can be described by an uncertain differential equation

$$dV_t = eV_t dt + \sigma V_t dC_t$$

where $e$ is the expected rate of return on the firm, $\sigma^2$ is the variance of the return on the firm, and $C_t$ is a canonical process. Now, we consider a simple case of corporate debt pricing. In this bond issue, the firm promises to pay a total of $B$ dollars to the bondholders on an expiration time $T$. If this payment is not met, the bondholders immediately take over the company. Clearly, if at time $T$, $V(T) > B$, the company should pay the promised payment $B$. If not, the bondholder will get $V_T$. Then the final payoff of bondholders is $\min(V_T, B)$. It is obvious that

$$\min(V_T, B) = B - \max(B - V_T, 0).$$

The payoff of the vulnerable corporate debt is $B$ minus a European put option with the strike price $B$, the stock price $V_t$ and the expiration time $T$. Next, we will calculate the value of this vulnerable corporate debt

$$f = E[\exp(-rT) \min(V_T, B)].$$
Theorem 1 Assume a corporate debt for the firm model has a strike price $K$ and an expiration time $T$. Then the value of the corporate debt is

$$f = \exp(-rT)B - \exp(-rT)V_0 \int_0^{K/V_0} \left( 1 + \exp \left( \frac{\pi(es - \ln y)}{\sqrt{3} \sigma T} \right) \right) dy.$$

Proof: It follows from the definition of $f$ that

$$f = E[\exp(-rT) \min(V_T, B)]$$

$$= \exp(-rT)B - E[\exp(-rT) \max(B - V_T, 0)]$$

$$= \exp(-rT)B - \exp(-rT)V_0 \int_0^{K/V_0} \left( 1 + \exp \left( \frac{\pi(es - \ln y)}{\sqrt{3} \sigma T} \right) \right) dy.$$

The corporate debt pricing formula with default risk is verified.

It is obvious that the value of the vulnerable corporate debt $f$ is decreasing with respect to the value of the firm $V_0$.

4 European Vulnerable Call Options

A European call option is a contract that gives the holder the right to buy a stock at the expiration time $T$ for a strike price $K$. Consider Liu’s stock model, we assume that a European call option has strike price $K$ and expiration time $T$. If $Y_t$ is the price of the underlying stock, then it is clear that the payoff from an European call option is $(Y_t - K)^+$ and the present value is

$$\exp(-rt)(Y_t - K)^+. \quad (2)$$

And the European call option price should be the expected present value of the payoff. Then this option has price

$$f_c = E \left[ \exp(-rt)(Y_t - K)^+ \right]. \quad (3)$$

The underlying asset for the option has value $Y_t$ at time $t$. In Liu [12], he proposed this model and gave the formula of European call options

$$f_c = \exp(-rT)V_0 \int_0^{\infty} \left( 1 + \exp \left( \frac{\pi(\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} dy.$$

Now, we consider European vulnerable call options. Suppose further that the value of a firm is $V_t$. At the option expiration time $T$, the option writer pays $\max(Y_T - K, 0)$ if it is solvent. If the option writer can not make the payment, the option holder gets the assets of the option writer $V_T$ at time $T$. Then the payoff of the option holder is $\min(\max(Y_T - K, 0), V_T)$. It is obvious that

$$\min(\max(Y_T - K, 0), V_T) = \max(\min(K + V_T, Y_T) - K, 0).$$

In fact that the European vulnerable call options is equal to an ordinary European call options with the stock price $\min(K + V_T, Y_T)$ and the unchanged strike price $K$. The value of the European vulnerable call options is

$$f_c = E[\min(\max(Y_T - K, 0), V_T)].$$

For the convenience of solving the model, we assume the firm value is a constant $V_t = V_0$, $0 \leq t \leq T$. Then we get the following theorem.

Theorem 2 Assume a European vulnerable call option for the stock model has a strike price $K$ and an expiration time $T$. Then the European vulnerable call option price is

$$f_c = \exp(-rT) \left( V_0 \wedge Y_0 \int_0^{\infty} \left( 1 + \exp \left( \frac{\pi(\ln y - eT)}{\sqrt{3} \sigma T} \right) \right)^{-1} dy \right).$$
Proof: It follows from the definition of $f_c$ that

$$f_c = E[\min(\max(Y_T - K, 0), V_T)]$$

$$= \exp(-rT)V_0 \land E[\exp(-rT) \max(Y_T - K, 0)]$$

$$= \exp(-rT) \left( V_0 \land Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi \ln y - eT}{\sqrt{3}\sigma T} \right) \right)^{-1} dy \right).$$

The European vulnerable call options formula with default risk is verified.

It is obvious that the European vulnerable call option price $f_c$ is increasing with respect to the value of the firm $V_0$.

5 European Vulnerable Put Options

A European put option is a contract that gives the holder the right to sell a stock at the expiration time $T$ for a strike price $K$. Consider Liu’s stock model, we assume that a European call option has strike price $K$ and expiration time $T$. If $Y_t$ is the price of the underlying stock, then it is clear that the payoff from an European put option is $(K - Y_t)^+$ and the present value is

$$\exp(-rt)(K - Y_t)^+.$$ (4)

And the European call option price should be the expected present value of the payoff. Then this option has price

$$f_c = E \left[ \exp(-rt)(K - Y_t)^+ \right].$$ (5)

The underlying asset for the option has value $Y_t$ at time $t$. In Liu [12], he proposed this model and gave the formula of European call options

$$f_p = \exp(-rT)V_0 \int_{0}^{K/Y_0} \left( 1 + \exp \left( \frac{\pi \ln y - \ln (t - T)}{\sqrt{3}\sigma T} \right) \right) dy.$$

Now, we consider European vulnerable put options. Suppose further that the value of a firm is $V_t$ At the option expiration time $T$, the option writer pays max($K - Y_T$, 0) if it is solvent. If the option writer cannot make the payment, the option holder get the assets of the option writer $V_T$ at time $T$. Then the payoff of the option holder is min(max($K - Y_T$, 0), $V_T$). It is obvious that

$$\min(\max(K - Y_T, 0), V_T) = \max(K - \min(K - Y_T, 0), V_T).$$

In fact, the European vulnerable put option is equal to an ordinary European put option with the stock price min($K - V_T$, $Y_T$) and the unchanged strike price $K$. The value of the European vulnerable call options is

$$f_p = E[\min(\max(K - Y_T, 0), V_T)].$$

For the convenience of solving the model, we assume the firm value is a constant $V_t = V_0$, $0 \leq t \leq T$.

Theorem 3 Assume a European vulnerable call option for the stock model has a strike price $K$ and an expiration time $T$. Then the European vulnerable call option price is

$$f_p = \exp(-rT) \left( V_0 \land Y_0 \int_{0}^{K/Y_0} \left( 1 + \exp \left( \frac{\pi \ln y - \ln (t - T)}{\sqrt{3}\sigma T} \right) \right) dy \right).$$

Proof: It follows from the definition of $f_p$ that

$$f_p = E[\min(\max(Y_T - K, 0), V_T)]$$

$$= \exp(-rT)V_0 \land E[\exp(-rT) \max(Y_T - K, 0)]$$

$$= \exp(-rT) \left( V_0 \land Y_0 \int_{K/Y_0}^{+\infty} \left( 1 + \exp \left( \frac{\pi \ln y - eT}{\sqrt{3}\sigma T} \right) \right)^{-1} dy \right).$$

The European vulnerable put options formula with default risk is verified.

It is obvious that the European vulnerable call option price $f_p$ is increasing with respect to the value of the firm $V_0$. 

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6 Conclusion

In this paper, we investigated the vulnerable option pricing problems for uncertain financial market. First, the corporate debt pricing with default risk was built. Then, European vulnerable call and put option models were proposed for Liu’s stock model. Some mathematical properties of these formulas were studied.

Acknowledgments

This work was supported by National Natural Science Foundation of China Grants No.70833003 and 91024032.

References


