Mean-TVaR Model for Portfolio Selection with Uncertain Returns

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Abstract

The mean-variance model proposed by Markowitz has received greatly acceptance as a practical methodology to manage portfolio selection, and has been widely extended in a variety of literatures. The aim of this paper is to extend the mean-variance model in uncertain decision systems. We present a new mean-TVaR model for portfolio selection when the returns of securities are described as uncertain variables. When the returns of the securities are characterized as some special uncertain variables such as linear uncertain variables, zigzag uncertain variables and normal uncertain variables, we employ the formulas of mean and TVaR to turn the mean-TVaR model to its equivalent problem. Since the crisp equivalent model is a linear programming, it can be solved by some convenient optimization algorithms such as interior point algorithm and simplex algorithm, or directly by some optimization software. Finally, we present a portfolio selection problem of fermenting foods to demonstrate the modeling idea and the effectiveness of the method.

Keywords: Tail value at risk (TVaR), Mean-TVaR model, Portfolio selection, Uncertain variable, Fermenting food

1 Introduction

Portfolio selection is concerned with optimization of capital allocation to a large number of securities. The most salient feature of security returns is uncertainty. Thus, how to describe the uncertain return and handle the risk brought about by the uncertainty is the key problem of portfolio selection. Generally, security returns are assumed to be random variables, and probability theory becomes the foundation for handling uncertainty. Under the assumption that the security returns are random variables, the mean-variance model proposed by Markowitz [1] has been widely extended, including mean-semivariance model [2], mean-absolute-deviation model [3], mean-variance-skewness model [4], mean-VaR model [5], and mean-CVaR model [6] proposed by Rockafellar and Uryaser in 2000. In risk management, CVaR is a desired risk measure by investors because of its property of coherence.

The basic assumption for using probability theory to manage portfolio selection is that the situation of security markets in future can be correctly reflected by historical data in the past, that is, the distributions of the uncertainty returns are similar to the past ones. However, it is difficult to ensure this kind of assumption for real ever-changing security markets. With the development of fuzzy set theory [7] proposed by Zadeh in 1965, many researchers realized the importance of handling possibilistic uncertainty in decision systems, and applied fuzzy set theory to portfolio optimization problems. Much work has been done on extending the mean-variance selection idea to fuzzy environment in different ways. For example, based on fuzzy chance theory, Huang [8] formulated a fuzzy chance-constrained model for portfolio selection. Gao [9] proposed credibilistic game with fuzzy information, and Liang et. al. [10] extended it to credibilistic strategic game. Zhang [11] discussed possibilistic mean-standard deviation models for bounded assets under the assumption that the returns of risky assets were fuzzy numbers. Li et. al. [12] constructed mean-variance-skewness model for portfolio selection with fuzzy returns. Bermudez [13] studied fuzzy portfolio selection with cardinality constrained. Zhang [14] proposed CVaR of fuzzy variable and constructed mean-CVaR model when the returns of security were assumed to be fuzzy variables.

In the above mathematical programming methods for dealing with portfolio selection problems, randomness and fuzziness are regarded as to be the two basic types of uncertainty contained in security returns. It used to be proposed that when security returns cannot be well reflected by historical data, we can use fuzzy variable to reflect experts’ knowledge and estimation of the security returns. However, a deeper research shows that paradoxes will appear if we use fuzzy variable to describe the subjective estimation of security returns.
For example, if a security return is regarded as a fuzzy variable, suppose that it is a triangular fuzzy variable 
\[ \xi \sim (-0.2, 0.5, 1.2) . \] It follows from possibility theory that the return being “exactly 0.5” and “not 0.5” are both with belief degree 1 in possibility measure. However, this conclusion is unacceptable because the return being “exactly 0.5” or not “exactly 0.5” have the same belief degree in possibility measure, which implies that the two events will happen equally likely. It seems that no human being can accept this conclusion. In 2007, Liu [15] proposed uncertain theory, which has become a branch of axiomatic mathematics for modeling human uncertainty. Liu [16] constructed uncertain Brownian motion and uncertain differential equations. Gao [17] discussed some properties in uncertainty theory when uncertain measure is continuous. Chen [18] presented some methods to solve linear uncertain differential equations, and proved an existence and uniqueness theorem of solution for uncertain differential equation under Lipschitz condition and linear growth condition. Zhang [19] presented some convergence concepts of uncertain sequence and the relationships among the convergence concepts were investigated. Recently, Peng [20] proved a sufficient and necessary condition of uncertain measure. Zhang [21] discussed uncertain programming model for project scheduling problem. Huang [22, 23, 24] described the return rates of securities as uncertain variables and proposed optimal portfolio selection models under different risk control. Yan [25] studied uncertain portfolio selection problem, a novel concept of VaR (value at risk) of uncertain loss was proposed to represent the investment risk and mean-VaR model for portfolio selection was constructed. As the counterpart of stochastic setting, though VaR is a measure of the risk of loss on a specific portfolio of financial assets in financial mathematics and risk management, however, a disadvantage of VaR is that it does not give any information about the severity of losses beyond the VaR value. Tail value at risk (for short, TVaR) is considered to provide a better measure of risk since it is able to quantity danger beyond value-at-risk and moreover it is coherent when independence is satisfied. In this paper, the returns are also described as uncertain variables, tail value at risk (TVaR) for the uncertain loss is used to represent the investment risk.

The rest of the paper is organized as follows. After recalling some necessary definitions and properties about uncertain variable in Section 2, mean-TVaR model for portfolio selection is proposed in Section 3. Then in Section 4, crisp equivalence of the mean-TVaR portfolio selection model is discussed when the returns are chosen as some special uncertain variables such as linear uncertain variable, zigzag uncertain variable and normal uncertain variable. In Section 5, a numerical example is provided to demonstrate the potential application and the effectiveness of the model. Finally, we conclude the paper in Section 6.

2 Preliminaries

Uncertainty theory was proposed by Liu [15] in 2007, and refined by Liu [26] in 2010. To better understand the proposed model for portfolio selection, we begin this section with reviewing some necessary knowledge about the theory.

Let \( \Gamma \) be a nonempty set, \( L \) be \( \sigma \)-algebra of a collection of subsets of \( \Gamma \). Let \( M \) be a nonnegative function from \( L \) to \([0, 1]\) satisfying normality axiom, duality axiom, subadditivity axiom and product axiom. Then \( M \) is called an uncertain measure, and the triple \((\Gamma, L, M)\) is called an uncertainty space.

Definition 1. ([26]) An uncertain variable \( \xi \) is a measurable function from an uncertainty space \((\Gamma, L, M)\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set 
\[ \{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \} \]
is an event.

Definition 2. ([26]) The uncertain variables \( \xi_1, \xi_2, \cdots, \xi_m \) are said to be independent if 
\[ M\left\{ \bigcap_{i=1}^{m} \{ \xi_i \in B_i \} \right\} = \min_{1 \leq i \leq m} M\{ \xi_i \in B_i \} \]
for any Borel sets \( B_1, B_2, \cdots, B_m \) of real numbers.

Just as a random variable can be characterized by a probability distribution function and a fuzzy variable may be described by a fuzzy membership function, an uncertain variable can be characterized by an uncertainty distribution function as follows.

Definition 3. ([26]) The uncertainty distribution \( \Phi(x) \) of an uncertain variable \( \xi \) is defined as 
\[ \Phi(x) = M\{ \xi \leq x \} \]
for any real number $x$.

For example, by a linear uncertain variable $\xi$, we mean the variable which has the following uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & x < a \\
\frac{x-a}{b-a}, & a \leq x \leq b \\
1, & x > b
\end{cases}
$$

where $a$ and $b$ are real numbers and $a < b$. For convenience, a linear variable $\xi$ with parameters $a$ and $b$ is denoted by $\xi \sim U(a, b)$.

If an uncertain variable $\xi$ has the following uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & x < a \\
\frac{x-a}{2(b-a)}, & a \leq x \leq b \\
\frac{x-c-2b}{2(c-b)}, & b \leq x \leq c \\
1, & x \geq c
\end{cases}
$$

where $a, b$ and $c$ are real numbers and with $a < b < c$. Then it is called a zigzag uncertain variable and denoted by $\xi \sim Z(a, b, c)$.

If an uncertain variable $\xi$ has the following normal uncertainty distribution

$$
\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}
$$

where $e$ and $\sigma$ are real numbers and $\sigma > 0$. Then it is called a normal uncertain variable and denoted by $\xi \sim N(e, \sigma)$.

To measure the size of an uncertain variable, B. Liu proposed the expected value of an uncertain variable as follows.

**Definition 4.** ([26]) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined as

$$
E[\xi] = \int_{0}^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^{0} M\{\xi \leq r\} dr
$$

provided that at least one of the two integrals is finite.

More generally, the expected value of an uncertain variable $\xi$ can be computed by its inverse distribution function as follows.

**Theorem 1.** ([26]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi(x)$. If the expected value of $\xi$ exists, then

$$
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha
$$

(1)

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty function of $\xi$ with respect to $\alpha$.

In order to measure finance risk and satisfy the practical demand, Peng [27] proposed notions of value-at-risk (for short, VaR) and tail value-at-risk (for short, TVaR) of uncertain variable as follows.

**Definition 5 ([27])** Let $\xi$ be an uncertain variable and $\alpha \in (0, 1]$ be the risk confidence level. Then the value-at-risk of $\xi$ at the confidence level $\alpha$, written as $\text{VaR}_\xi(\alpha)$, is the function $\text{VaR}_\xi(\alpha): (0, 1] \rightarrow \mathbb{R}$ such that

$$
\text{VaR}_\xi(\alpha) = \inf\{x | M\{\xi \leq x\} \geq \alpha\}.
$$

**Definition 6 ([27])** Let $\xi$ be an uncertain variable and $\alpha \in (0, 1]$ be the risk confidence level. Then tail value-at-risk of $\xi$ at the confidence level $\alpha$, written as $\text{TVaR}_\xi(\alpha)$, is the function $\text{TVaR}_\xi(\alpha): (0, 1] \rightarrow \mathbb{R}$ such that

$$
\text{TVaR}_\xi(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} \text{VaR}_\xi(\beta) d\beta.
$$

In [27], it has been proved that TVaR of an uncertain variable has properties of translation invariance, monotonicity, positive homogeneity, and subadditivity under independence. So TVaR is a coherent risk measure under the independence condition.

For more details about uncertainty theory, interested reader may consult [26, 27].
3 Mean-TVaR model for portfolio selection

To illustrate the approach we propose, we consider now the case in which the decision vector \( x \) represents a portfolio of securities in the sense that \( x = (x_1, x_2, \cdots, x_n)^T \) with \( x_i \) being the proportion in security \( i \) and \( x_i \geq 0 \) for \( i = 1, 2, \cdots, n \) with \( \sum_{i=1}^{n} x_i = 1 \). Denoting by \( \xi_i \) the uncertain return on security \( i \), we take the uncertain vector \( \xi \) to be \( \xi = (\xi_1, \xi_2, \cdots, \xi_n)^T \).

The total return on a portfolio \( x \) is the sum of the returns on the individual securities in the portfolio, scaled by the proportions \( x_i \), i.e., \( x^T \xi = \sum_{i=1}^{n} x_i \xi_i \). The loss, being the negative of this, is given therefore by

\[
f(x, \xi) = -x^T \xi = -(x_1 \xi_1 + x_2 \xi_2 + \cdots + x_n \xi_n).
\]

Thus, VaR of the loss \( f(x, \xi) \) at the confidence level \( \alpha \) with respect to the portfolio \( x \) can be described as

\[
\text{VaR}_{f(x, \xi)}(\alpha) = \inf\{y | M\{f(x, \xi) \leq y\} \geq \alpha\},
\]

for \( \alpha \in (0, 1] \).

Furthermore, TVaR of the loss \( f(x, \xi) \) at the confidence level \( \alpha \) with respect to the portfolio \( x \) can be represented by

\[
\text{TVaR}_{f(x, \xi)}(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} \text{VaR}_{f(x, \xi)}(\beta) d\beta
\]

for \( \alpha \in (0, 1] \).

For simplicity, \( \text{VaR}_{f(x, \xi)}(\alpha) \) and \( \text{TVaR}_{f(x, \xi)}(\alpha) \) of the loss function \( f(x, \xi) \) are denoted by \( \text{VaR}(\alpha) \) and \( \text{TVaR}(\alpha) \), respectively, without declaration again in the rest of this paper.

We restrict the uncertain vector \( \xi \) to the continuous type below, and denote the distribution function of the total return \( \sum_{i=1}^{n} x_i \xi_i \) by \( \Phi \), i.e.,

\[
\Phi(y) = M\{\sum_{i=1}^{n} x_i \xi_i \leq y\}.
\]

Then \( \text{VaR}(\alpha) \) of the loss function \( f(x, \xi) \) can be formatted as follows,

\[
\text{VaR}_{f(x, \xi)}(\alpha) = \inf\{y | M\{f(x, \xi) \leq y\} \geq \alpha\} = \inf\{y | M\{-\sum_{i=1}^{n} x_i \xi_i \leq y\} \geq \alpha\} = \inf\{y | 1 - M\{-\sum_{i=1}^{n} x_i \xi_i \leq -y\} \geq \alpha\} = \inf\{y | M\{\sum_{i=1}^{n} x_i \xi_i \leq -y\} \leq 1 - \alpha\} = -\Phi^{-1}(1 - \alpha).
\]

TVaR(\alpha) of the loss function \( f(x, \xi) \) thus can be computed as follows,

\[
\text{TVaR}(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} -\Phi^{-1}(1 - \beta) d\beta.
\]

(2)

To obtain the maximum investment return and avoid possible risk, an investor should select an optimal portfolio. If the investor desires to set a goal of maximizing the expected return of a portfolio, and requires that the investment risk is not greater than some preset risk level \( \delta \), then the mathematical expression of the selection idea can be summarized as follows,

\[
\begin{align*}
\text{max} & \quad E[x^T \xi] \\
\text{s.t.} & \quad \text{TVaR}(\alpha) \leq \delta, \\
& \quad \sum_{i=1}^{n} x_i = 1, \\
& \quad x_i \geq 0, \quad i = 1, 2, \cdots, n.
\end{align*}
\]

(3)
where \( E \) is the expected value of the total return, \( \text{TVaR}(\alpha) \) is the tail value at risk of the loss \( f(x, \xi) \) at confidence level \( \alpha \), the parameter \( \delta \) is the upper risk level that the investor can tolerate. For convenience, model (3) is called mean-TVaR model for portfolio selection. It should be pointed out that the confidence level \( \alpha \) in model (3) is usually selected as close to 1 in practice, such as \( \alpha = 0.7, 0.8, 0.85, 0.9, 0.95 \) and so on.

### 4 Crisp equivalence of mean-TVaR model

In this section, we will discuss the crisp equivalence of model (3) when the returns \( \xi_1, \xi_2, \cdots, \xi_n \) are chosen as some special uncertain variables such as linear uncertain variables, zigzag uncertain variables and normal uncertain variables.

**Theorem 2** [26] Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If \( f(x_1, x_2, \cdots, x_n) \) is a strictly increasing function of \( x_1, x_2, \cdots, x_n \), then \( f(\xi_1, \xi_2, \cdots, \xi_n) \) is an uncertain variable with inverse uncertainty distribution

\[
\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).
\]

**Corollary 1** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. Then the inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) of \( \sum_{i=1}^{n} x_i \xi_i \) is

\[
\Phi^{-1}(\alpha) = x_1 \Phi_1^{-1}(\alpha) + x_2 \Phi_2^{-1}(\alpha) + \cdots + x_n \Phi_n^{-1}(\alpha),
\]

where \( x_i \geq 0, i = 1, 2, \cdots, n \).

In the sequel, we will discuss the equivalent formulations of model (3) for the following three special cases.

**Case 1**

Assume that the returns \( \xi_1, \xi_2, \cdots, \xi_n \) are independent linear uncertain variables, denoted by

\[
\xi_i \sim U(a_i, b_i), i = 1, 2, \cdots, n.
\]

Then the inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) of the total return \( \sum_{i=1}^{n} x_i \xi_i \) is

\[
\Phi^{-1}(\alpha) = \sum_{i=1}^{n} x_i \Phi_i^{-1}(\alpha) = \sum_{i=1}^{n} x_i a_i + \alpha \sum_{i=1}^{n} x_i (b_i - a_i), \quad \alpha \in (0, 1).
\]

It follows from (1) that the expected value of the total return is

\[
E \left[ \sum_{i=1}^{n} x_i \xi_i \right] = \int_0^1 \Phi^{-1}(\alpha) d\alpha = \sum_{i=1}^{n} \frac{x_i (a_i + b_i)}{2}.
\]

In accordance with (2), the \( \text{TVaR}(\alpha) \) of the loss function \( - \sum_{i=1}^{n} x_i \xi_i \) is

\[
\text{TVaR}(\alpha) = \frac{1}{1 - \alpha} \int_0^1 -\Phi^{-1}(1 - \beta) d\beta = \sum_{i=1}^{n} x_i \left( \frac{-1 - a_i}{2} - \frac{1 - \alpha}{2} b_i \right), \quad \alpha \in (0, 1).
\]

Therefore, model (3) can be equivalently written as follows,

\[
\begin{align*}
\text{max} & \sum_{i=1}^{n} x_i (a_i + b_i) \\
\text{s.t.} & \sum_{i=1}^{n} x_i [(-1 - \alpha) a_i - (1 - \alpha) b_i] \leq 2\delta, \\
& \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0, i = 1, 2, \cdots, n.
\end{align*}
\]

\[\text{MEAN-TVaR MODEL FOR PORTFOLIO SELECTION WITH UNCERTAIN RETURNS}\]
Case 2
Assume that the returns $\xi_1, \xi_2, \cdots, \xi_n$ are independent zigzag uncertain variables, denoted by

$$\xi_i \sim Z(a_i, b_i, c_i), \ a_i < b_i < c_i, \ i = 1, 2, \cdots, n.$$ 

Then the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ of the total return $\sum_{i=1}^{n} x_i \xi_i$ is

$$\Phi^{-1}(\alpha) = \sum_{i=1}^{n} x_i \Phi^{-1}_i(\alpha) = \begin{cases} \sum_{i=1}^{n} x_i a_i + 2\alpha \sum_{i=1}^{n} x_i (b_i - a_i), & \alpha \in (0, 0.5), \\ 2 \sum_{i=1}^{n} x_i b_i - \sum_{i=1}^{n} x_i c_i + 2\alpha \sum_{i=1}^{n} x_i (c_i - b_i), & \alpha \in [0.5, 1). \end{cases}$$

Therefore, the expected value of the total return $\sum_{i=1}^{n} x_i \xi_i$ is

$$E\left[\sum_{i=1}^{n} x_i \xi_i\right] = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha = \sum_{i=1}^{n} x_i \left(\frac{a_i + 2b_i + c_i}{4}\right).$$

In this case, the TVaR($\alpha$) of the loss function $-\sum_{i=1}^{n} x_i \xi_i$ is

$$\text{TVaR}(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} -\Phi^{-1}(1-\beta) d\beta = \begin{cases} \sum_{i=1}^{n} x_i[-\frac{1}{4}a_i + (a^2 - \frac{1}{2})b_i + (-a^2 + a - \frac{1}{4})c_i], & \alpha \in (0, 0.5), \\ \sum_{i=1}^{n} x_i[-\alpha a_i - (1-\alpha)b_i], & \alpha \in [0.5, 1). \end{cases}$$

When the confidence level $\alpha$ is chosen as greater than 0.5, that is, $\alpha$ belongs to the interval (0.5, 1), model (3) can be equivalently written as follows,

$$\begin{align*}
\max & \sum_{i=1}^{n} x_i (a_i + 2b_i + c_i) \\
\text{s.t.} & \sum_{i=1}^{n} x_i [-\alpha a_i - (1-\alpha)b_i] \leq \delta, \\
& \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0, i = 1, 2, \cdots, n.
\end{align*}$$

Case 3
Assume that the returns $\xi_1, \xi_2, \cdots, \xi_n$ are independent normal uncertain variables, denoted by

$$\xi_i \sim N(e_i, \sigma_i), \ \sigma_i > 0, \ i = 1, 2, \cdots, n.$$ 

Then the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ of $\sum_{i=1}^{n} x_i \xi_i$ is

$$\Phi^{-1}(\alpha) = \sum_{i=1}^{n} x_i \Phi^{-1}_i(\alpha) = \sum_{i=1}^{n} x_i \left(e_i + \frac{\sigma_i \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)$$

where $x_i \geq 0, i = 1, 2, \cdots, n$ and $\alpha \in (0, 1)$.

The expected value of the total return $\sum_{i=1}^{n} x_i \xi_i$ is

$$E\left[\sum_{i=1}^{n} x_i \xi_i\right] = \int_{0}^{1} \left[\sum_{i=1}^{n} x_i \left(e_i + \frac{\sigma_i \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)\right] d\alpha = \sum_{i=1}^{n} x_i e_i.$$
In this case, the TVaR(\(\alpha\)) of the loss function \(-\sum_{i=1}^{n} x_i \xi_i\) is:

\[
TVaR(\alpha) = \frac{1}{1 - \alpha} \int_{-\infty}^{\alpha} -\Phi^{-1}(1 - \beta)d\beta
\]

\[
= -\sum_{i=1}^{n} x_i e_i + \frac{\sqrt{3}}{\pi} \left[ -\ln(1 - \alpha) - \frac{\alpha}{1 - \alpha} \ln \alpha \right] \sum_{i=1}^{n} x_i \sigma_i
\]

\[
= \sum_{i=1}^{n} x_i \left\{ -e_i + \frac{\sqrt{3}}{\pi} \left[ -\ln(1 - \alpha) - \frac{\alpha}{1 - \alpha} \ln \alpha \right] \sigma_i \right\}
\]

Therefore, model (3) can be equivalently written as follows,

\[
\text{max} \quad \sum_{i=1}^{n} x_i e_i
\]

s.t.

\[
\sum_{i=1}^{n} x_i \left\{ -e_i + \frac{\sqrt{3}}{\pi} \left[ -\ln(1 - \alpha) - \frac{\alpha}{1 - \alpha} \ln \alpha \right] \sigma_i \right\} \leq \delta
\]

\[
\sum_{i=1}^{n} x_i = 1,
\]

\[
x_i \geq 0, \quad i = 1, 2, \cdots, n.
\]

It is apparent that models (4)-(6) are linear programming. Thus we can use conventional optimization algorithms such as interior point and simplex method to obtain their optimal solutions.

### 5 An example

To illustrate applications of mean-TVaR model and the modeling idea, we present one numerical example as follows in this section.

**Example 1.** According to National Bureau of Statistics of China report on February 12, 2011, price of most majority of 29 kinds of primary foods in 50 large and median-sized cities continued to increase as before. This strongly resulted in the rapid increment of return rates of securities of fermenting foods such as Xiangjigui, Guizhomaotai, Wuliangye, Luzhoulaojiao and so on. Suppose that an investor wants to invest before. This strongly resulted in the rapid increment of return rates of securities of fermenting foods such as Xiangjigui, Guizhomaotai, Wuliangye, Luzhoulaojiao and so on. Suppose that an investor wants to invest his/her money to several securities newly registered in 2012. Being short of historical data about the return rates of the chosen securities, it is necessary to invite some domain experts to evaluate the return rates of the chosen securities, thus the uncertain return rates are given by experts, which are indicated in Table 1. The parameters \(\alpha\) and \(\delta\) in model (6) are assigned to 0.7 and -0.1 by the investor, respectively.

<table>
<thead>
<tr>
<th>Security</th>
<th>Return rates (\xi_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\xi_1 \sim N(0.5, 0.3))</td>
</tr>
<tr>
<td>2</td>
<td>(\xi_2 \sim N(0.6, 0.4))</td>
</tr>
<tr>
<td>3</td>
<td>(\xi_3 \sim N(0.7, 0.5))</td>
</tr>
<tr>
<td>4</td>
<td>(\xi_4 \sim N(0.7, 0.6))</td>
</tr>
<tr>
<td>5</td>
<td>(\xi_5 \sim N(0.8, 0.7))</td>
</tr>
<tr>
<td>6</td>
<td>(\xi_6 \sim N(0.9, 0.75))</td>
</tr>
</tbody>
</table>

According to the above discussion, we have

\[
E[x^T \xi] = 0.5x_1 + 0.6x_2 + 0.7x_3 + 0.7x_4 + 0.8x_5 + 0.9x_6,
\]

\[
TVaR(\alpha) = x_1(-0.5 - 0.3 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha)) + x_2(-0.6 - 0.4 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha))
\]

\[
= x_3(-0.7 - 0.5 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha)) + x_4(-0.7 - 0.6 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha))
\]

\[
= x_5(-0.8 - 0.7 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha)) + x_6(-0.9 - 0.75 \times \frac{\sqrt{3}}{\pi} (\ln(1 - \alpha) + \frac{\alpha}{1 - \alpha} \ln \alpha))
\]

\[
= -0.16321x_1 - 0.15095x_2 - 0.13869x_3 - 0.02643x_4 - 0.01416x_5 - 0.05803x_6
\]
Thus, model (6) can be transformed into the following form,

$$\begin{align*}
\text{max} & \quad 0.5x_1 + 0.6x_2 + 0.7x_3 + 0.7x_4 + 0.8x_5 + 0.9x_6 \\
\text{s.t.} & \quad -0.16321x_1 - 0.15095x_2 - 0.13869x_3 - 0.02643x_4 - 0.01416x_5 - 0.05803x_6 \leq -0.1, \\
& \quad \sum_{i=1}^{6} x_i = 1, \\
& \quad x_i \geq 0, i = 1, 2, 3, 4, 5, 6. 
\end{align*}$$

(7)

The global optimal solution to model (7) is listed in Table 2.

<table>
<thead>
<tr>
<th>Security</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allocation of money</td>
<td>0</td>
<td>0</td>
<td>52.03323</td>
<td>0</td>
<td>0</td>
<td>47.96677</td>
</tr>
</tbody>
</table>

The objective function value of model (7) is 0.79593, that is, the expected return of the optimal portfolio selection of these 6 securities of fermenting foods is approximately 0.79593.

**Remark 1** The result of model (7) is performed on a personal computer by taking advantage of the software LINGO.

**Remark 2** The investor can select different values of the parameters $\alpha$ and $\delta$ according to his/her interest.

### 6 Conclusion

In this paper, we have discussed a portfolio selection problem in which security returns are described as uncertain variables. A new mean-TVaR model for portfolio selection has been constructed. The computational formulas of the expected value of the total return and TVaR of the loss function are provided. In order to solve the proposed model, the crisp equivalences of the model are considered when security returns are chosen as some special uncertain variables such as linear uncertain variables, zigzag uncertain variables and normal uncertain variables, thus the classical optimization algorithms can be used. Several further issues should be considered. For example, usually there isn’t closed-form of the expected value of total return and TVaR of loss function when security returns are general uncertain variables, hybrid intelligent algorithm must be designed to solve the model in this case.

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### References


