Uncertain Optimal Control with Application to Vehicle Routing Problem

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Abstract

This paper presents a vehicle routing model for uncertain vehicle routing problem by using uncertainty theory. On the basis of mathematical programming method and uncertainty theory, some optimal routing formulas on the proposed uncertain vehicle routing model are investigated and some numerical examples are illustrated, which may be helpful to investigate complicated vehicle routing system.

Keywords: Uncertainty theory; Optimal control; Vehicle routing problem

1 Introduction

Since the 1950s, the optimal control theory has been studied widely as an important branch of modern control theory. The study of optimal control greatly attracted the attention of many mathematicians because of the necessity of strict expression form in optimal control theory. With the greater use of methods and results in mathematics and computer science, optimal control theory has successfully been applied to diverse fields such as motor control theory [1, 2], epidemiology [3], bioscience [4] and launch vehicle control [5].

There are a number of situations where different events exhibit the complexity of the world. Therefore, Liu introduced an uncertain measure to measure the truth degree of an uncertain event [6]. Based on the uncertain measure, uncertainty theory was founded by Liu in 2007 and subsequently studied by many researchers. Until now, uncertainty theory has become a branch of axiomatic mathematics for investigating the behavior of uncertain phenomena. It has successfully been applied to such diverse fields of interest as economics [7-9], transportation [10], and human language [11].

Vehicle routing problem is concerned with optimal routes control, beginning times and fleet of vehicles. Due to its wide applicability and economic importance, vehicle routing problem has been extensively studied. Liu [12] first introduced uncertainty theory into the research area of vehicle routing problem in 2009. From then on, uncertainty theory has provided a new approach to deal with nondeterministic factors in vehicle routing problem. Obviously, the travelling time of each path in an uncertain vehicle routing problem is uncertain, and we cannot get the shortest path in the normal sense. Therefore, we study the vehicle routing control problem using uncertainty theory in this paper. In detail, vehicle routing problem will be modelled by uncertain programming in which the travel times are assumed to be uncertain variables with known uncertainty distributions.

The organization of this paper is as follows. Firstly, the uncertain measure, uncertainty distribution and their theorems are given in Section 2. And then, the uncertain optimal control technique is proposed in Section 3. Finally, important conclusions drawn from this study are provided in Section 4.

2 Preliminary

For convenience, we introduce some concepts in this section. Let \( C \) be a nonempty set and \( L \) be a \( \sigma \)-algebra over \( C \). Each element \( \Lambda \) in \( \sigma \)-algebra \( L \) is called an event.

**Definition 2.1**[6, 13] The set function \( M \) is called an uncertain measure if it satisfies the following axioms:

- Axiom 1 (normality): \( M\{C\} = 1 \) for the universal set \( C \);
- Axiom 2 (duality): \( M\{\Lambda\} + M\{\Lambda^C\} = 1 \) for any event \( \Lambda \);
Axiom 3 (subadditivity): \( M \big\{ \bigcup_{i=1}^{\infty} \Lambda_i \big\} \leq \sum_{i=1}^{\infty} M \{ \Lambda_i \} \) for every countable sequence of events \( \Lambda_i \).

Axiom 4 (product): Let \( (C_k, L_k, M_k) \) be uncertainty spaces for \( k = 1, 2, \cdots \). The product uncertain measure \( M \) is an uncertain measure satisfying \( M \{ \prod_{i=1}^{\infty} \Lambda_i \} = \bigwedge_i (M \{ \Lambda_i \}) \), where \( \Lambda_i \) are arbitrarily chosen events from \( L_k \) for \( k = 1, 2, \cdots \), respectively.

Definition 2.2[6] The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by
\[
\Phi(x) = M \{ \xi \leq x \}
\]
for any real \( x \).

Obviously, the uncertainty distribution gives a description of uncertain variables. In many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself. Some useful uncertainty distributions are given here.

Example 2.3[6] An uncertain variable \( \xi \) is called linear if it has a linear uncertainty distribution
\[
\Phi(x) = \begin{cases}
0, & x \leq a \\
\frac{x-a}{b-a}, & a \leq x \leq b \\
1, & x \geq b
\end{cases}
\]
denoted by \( L(a, b) \) where \( a \) and \( b \) are real numbers with \( a < b \).

Example 2.4[6] An uncertain variable \( \xi \) is called normal if it has a normal uncertainty distribution
\[
\Phi(x) = \left[ 1 + \exp \left( \frac{\pi(e-x)}{\sqrt{3} \pi} \right) \right]^{-1}, \quad x \in \mathbb{R}
\]
denoted by \( N(e, \sigma) \) where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \).

Liu presented the measure inverse theorem for uncertainty distribution[14].

Theorem 2.5 (Measure Inversion Theorem) [14] Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \). Then for any real number \( x \), we have
\[
M \{ \xi \leq x \} = \Phi(x), \quad M \{ \xi \geq x \} = 1 - \Phi(x).
\]

Theorem 2.6[14] Let \( \xi \) be an uncertain variable with continuous uncertainty distribution \( \Phi \). Then for any interval \([a, b]\), we have
\[
\Phi(b) - \Phi(a) \leq M \{a \leq \xi \leq b\} \leq \Phi(b) \wedge (1 - \Phi(a))
\]

Definition 2.7[6] An uncertainty distribution \( \Phi \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \).

Definition 2.8[6] Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). Then the inverse function \( \Phi^{-1} \) is called the inverse uncertainty distribution of \( \Phi \).

Note that the inverse uncertainty distribution \( \Phi^{-1}(\alpha) \) is well defined on the open interval \((0, 1)\). If needed, we may extend the domain to \([0, 1]\) via
\[
\Phi^{-1}(0) = \lim_{\alpha \to 0^+} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \to 1^-} \Phi^{-1}(\alpha).
\]

It is easy to verify that an inverse uncertainty distribution is a monotone increasing function on \([0, 1]\).

Example 2.9[6] The inverse uncertainty distribution of linear uncertain variable \( L(a, b) \) is
\[
\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.
\]

Example 2.10[6] The inverse uncertainty distribution of normal uncertain variable \( N(e, \sigma) \) is
\[
\Phi^{-1}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

Definition 2.11[6] Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by
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\[ E(\xi) = \int_{-\infty}^{0} M\{\xi \geq r\} dr - \int_{0}^{\infty} M\{\xi \leq r\} dr. \]

provided that at least one of the two integrals is finite.

**Theorem 2.12** \[6\] Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then

\[ E(\xi) = \int_{0}^{\infty} (1 - \Phi(x)) dx - \int_{-\infty}^{0} \Phi(x) dx. \]

**Theorem 2.13** \[6\] Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). If the expected value exists, then

\[ E(\xi) = \int_{0}^{\infty} \Phi^{-1}(\alpha) d\alpha. \]

Based on the uncertainty theory, Liu \[6\] introduced the concepts of uncertain programming which is a type of mathematical programming involving uncertain variables. In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu proposed the following uncertain programming model \[12\], \[(,\, ) \]

\[ \min \mathbb{E} f(x) \]

subject to:

\[ M\{g_j(x, \xi) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \]

And then, Liu gave a proof of the equivalent theorem \[6\].

**Theorem 2.14** \[6\] Assume \( f(x, \xi_1, \xi_2, \cdots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n \) and \( g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \) are strictly increasing with respect to \( \xi_1, \xi_2, \cdots, \xi_k \), and strictly decreasing with respect to \( \xi_{k+1}, \xi_{k+2}, \cdots, \xi_n \) for \( j = 1, 2, \cdots, p \). If \( \xi_1, \xi_2, \cdots, \xi_n \) are independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively, then the uncertain programming

\[
\begin{align*}
\min \mathbb{E} f(x, \xi_1, \xi_2, \cdots, \xi_n) \\
\text{subject to:} \\
M\{g_j(x, \xi_1, \xi_2, \cdots, \xi_n) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.
\end{align*}
\]

is equivalent to the crisp mathematical programming,

\[
\begin{align*}
\min \int_{0}^{1} f(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) d\alpha \\
\text{subject to:} \\
M\{g_j(x, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p.
\end{align*}
\]

### 3 Vehicle Routing Problem

An uncertain optimal control problem is to choose the best decision by optimizing some objective function related to an uncertain process. Because the objective function is an uncertain variable, for any decision, we cannot optimize it as a real function. Fortunately, an uncertain optimal control problem can be translated into a mathematical programming problem using Theorem 2.14.

Based on the study of vehicle routing problem with uncertain processing \[12\], this paper discusses routing optimal problem.

In a routing optimal problem, we first assume (i) a vehicle will be assigned for one and only one route on which there may be more than one customer; (ii) a customer will be visited by one and only one vehicle; (iii) each route begins and ends at the depot; (iv) each customer specifies its time window within which the delivery is permitted or preferred to start.

Let \( f_i(\bar{x}, \bar{y}, \bar{t}) \) be the arrival time functions of some vehicles at customers \( i \), for \( i = 1, 2, \cdots, n \), where
\( \bar{x} = (x_1, x_2, \cdots, x_n) \) is an integer decision vector representing \( n \) customers with \( 1 \leq x_i \leq n \) and \( x_i \neq x_j \) for all \( i \neq j \), \( i, j = 1, 2, \cdots, n, \) \( \vec{y} = (y_1, y_2, \cdots, y_{m-1}) \) integer decision vector with \( 0 = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq y_m = n \), \( \bar{t} = (t_1, t_2, \cdots, t_m) \) represents the starting times of vehicles \( k \) at the depot. Here we assume that the customer does not permit a delivery earlier than the time window. If a vehicle arrives at a customer after the beginning of the time window, unloading will start immediately. For each \( k \) with \( 1 \leq k \leq m \), if vehicle \( k \) is used, then we have

\[
f_{x_{k-1}^+}^1(x, y, t) = t_k + T_{0,x_{k-1}^+}
\]

and

\[
f_{x_{k-1}^+}^1(x, y, t) = f_{x_{k-1}^+}^1(x, y, t) \lor a_{x_{k-1}^+} + T_{x_{k-1}^+} + \tau_{x_{k-1}^+}
\]

where \( T_{i,j} \) are the uncertain travel times from customers \( i \) to \( j \), \( i, j = 0, 1, 2, \cdots, n \), \( \tau_k \) is the spent times for unloading and return of goods.

The inverse uncertainty distribution is

\[
\psi^{-1}_{x_{k-1}^+} (x, y, t, \alpha) = t_k + \Phi^{-1}_{0,x_{k-1}^+} (\alpha)
\]

and

\[
\psi^{-1}_{x_{k-1}^+} (x, y, t, \alpha) = \psi^{-1}_{x_{k-1}^+} (x, y, t, \alpha) \lor a_{x_{k-1}^+} + \Phi^{-1}_{x_{k-1}^+} + \gamma^{-1}_{x_{k-1}^+}
\]

where \( \Phi_{i,j} \) are the uncertainty distributions of \( T_{i,j} \) and \( \gamma_k \) are the uncertainty distributions of \( \tau_k \).

Let \( d(x, y) \) be the total travel distance of all vehicles. Then we have

\[
d(x, y) = \sum_{k=1}^{m} d_k(x, y)
\]

where

\[
d_k(x, y) = \begin{cases} 
D_{0,x_{k-1}^+} + \sum_{j=y_{k-1}^+}^{y_k} D_{x_j, x_{j+1}} + D_{x_{y_k}, 0} & \text{if } y_k > y_{k-1} \\
D_{x_{y_k-1}, x_{y_k}} & \text{if } y_k = y_{k-1}, k = 1, 2, \cdots, m
\end{cases}
\]

If we hope that each customer \( i \) is visited within its time window \([a_i, b_i]\) with confidence level \( \alpha_i \), then we have the following chance constraint,

\[
M \{ f_i(\bar{x}, \bar{y}, \bar{t}) \leq b_i \} \geq \alpha_i
\]

If we want to minimize the total travel distance of all vehicles subject to the time window constraint, then we have the following vehicle routing model,

\[
\begin{aligned}
\min d(x, y) \\
\text{subject to:}
\end{aligned}
\]

\[
\begin{aligned}
M \{ f_i(\bar{x}, \bar{y}, \bar{t}) \leq b_i \} \geq \alpha_i, & \quad i = 1, 2, \cdots, n \\
1 \leq x_i \leq n, & \quad i = 1, 2, \cdots, n \\
x_i \neq x_j, & \quad i \neq j, \quad i, j = 1, 2, \cdots, n \\
0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n \\
x_i, \ y_j, & \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1
\end{aligned}
\]

which is equivalent to
EXAMPLE

Assume that there are 2 vehicles and 4 customers with the time windows as follow:

<table>
<thead>
<tr>
<th>Customer</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Window</td>
<td>[13:00,15:00]</td>
<td>[13:00,15:00]</td>
<td>[19:00,21:00]</td>
<td>[19:00,21:00]</td>
</tr>
</tbody>
</table>

We suppose that each customer is visited within time windows with confidence level 0.90. We also assume that the travel distances from customers \( i \) to \( j \) are \( D_{ij} = |i-j|, i, j = 0, 1, \ldots, 4 \), and the travel times are normal uncertain variables \( T_{ij} \sim N(|i-j|,1) \), \( i, j = 0, 1, \ldots, 4 \).

Then the optimal solution is \( \bar{x}^* = [1,3;2,4]; \quad \bar{y}^* = [1,2]; \quad \bar{\tau}^* = [11:36,10:23] \).

In other words, the optimal operational plan is:

- Vehicle 1: depot \( \rightarrow 1 \rightarrow 3 \rightarrow \) depot, starting time: 11:36
- Vehicle 2: depot \( \rightarrow 2 \rightarrow 4 \rightarrow \) depot, starting time: 10:23

whose total travel distance is 14.

Generally speaking, we can obtain the shortest path in the uncertain vehicle routing problem by using the following three steps.

1. Step 1: Calculate \( \Psi^{-1}_i(\alpha) \), for the travelling time of vehicle \( i \).
2. Step 2: Construct a deterministic vehicle routing network.
3. Step 3: Employ the uncertain algorithm to get the shortest path in vehicle routing network.

The path we obtain in Step 3 is just the shortest path we want.

4 Conclusion

Applications of the ideas gained from uncertainty theory to optimize vehicle routing problem have been one of the most exciting areas of research in recent years. This paper presented an uncertain vehicle routing model, which is carried out by the uncertain programming method. In this paper, we are devoted to providing some optimal routing formulas. In addition, some numerical experiments were given to illustrate the formulas.

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References


