Uncertain Delayed Renewal Reward Process and Its Applications

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Abstract- Uncertain process is a sequence of uncertain variables indexed by time. This paper aims to introduce a kind of uncertain process named uncertain delayed renewal reward process whose interarrival times and rewards (or costs) are regarded as uncertain variables with the first interarrival (i.e., renewal) time and reward different from the others, respectively. The main results include the uncertainty distribution of delayed renewal reward process and two uncertain delayed renewal reward theorems on the limit value of reward rate. Finally, the application of the uncertain delayed renewal reward theorem are discussed and four examples are given.

Index Terms- Uncertain renewal reward process, uncertain delayed renewal reward process, uncertain process, uncertainty theory.

I. INTRODUCTION

Renewal process is a type of random process that generalizes Poisson processes for arbitrary malfunctions times. Renewal reward process was viewed as the successive repair costs incurred as a result of the successive malfunctions where interarrival times and rewards are regarded as random variables. A lot of researches have been down in renewal processes and renewal reward processes. Besides random renewal process, renewal process with fuzzy interarrival times and rewards has been studied by Zhao and Liu [22]. And renewal process with fuzzy random interarrival times was studied by Zhao and Tang [23]. Propova and Wu [?] studied a renewal reward process with random interarrival times and fuzzy random rewards. Hwang [5] investigated the renewal rate for the fuzzy random renewal process. Hong [3] studied a renewal process in which interarrival times and rewards were depicted by L-R fuzzy variables under triangular norm.

In practice, when we apply probability theory, a large samples are desired to estimate probability distribution. When no samples are available, we have to invite some domain experts to evaluate the degree of belief which each event will occur. When the degree of belief is considered as subjective
probability or fuzzy concept, it may lead to counterintuitive results. In order to rationally deal with
degrees of belief, an uncertainty theory was founded by Liu [8]. This theory has become a branch of
axiomatic mathematics for modeling human uncertainty.

In the framework of uncertainty theory, Liu [9] assumed the interarrival times of an renewal process
are uncertain variables, and proposed an uncertain renewal process. Then Liu [12] proposed an uncertain
renewal reward process whose interarrival times and rewards are both regarded as uncertain variables
and gave the elementary uncertain renewal theorem. Later, Liu [14] applied this theory to an uncertain
insurance model. Inspired by those achievements, Yao and Li [18] proposed an uncertain alternating
renewal process whose off-times and on-times are regarded as uncertain variables. Besides, Yao [17]
studied uncertain calculus of uncertain renewal process, and Yu [19] discussed a stock model with jumps
in an uncertain market. Recently, as a continuation of Liu’s work, Zhang et al. [21] proposed the so
called uncertain delayed renewal process whose first interarrival time is supposed to be different from
the others. The aim of this paper is to discuss the so-called uncertain delayed renewal reward process
of which the first interarrival time and the first reward are assumed to be different from the others,
respectively. In the paper, we first present a kind of uncertain process, named uncertain delayed renewal
reward process whose interarrival time and reward (or costs) are regarded as uncertain variables, and
whose first interarrival (i.e., renewal) time and reward are supposed to be different from the others,
respectively.

The rest of this paper is organized as follows. In Section 2, we recall some basic concepts and
results about uncertainty theory. In Section 3, we recall some basic concepts and results about uncertain
renewal process, uncertain renewal reward process, uncertain delayed renewal process. In Section 4, we
will introduce uncertain delayed renewal reward process and will study uncertain delayed renewal reward
theorems. In Section 5, we will provide several applied examples. Finally, we will give a brief summary.

II. PRELIMINARY

In this section, we recall some basic concepts and results about uncertainty theory and uncertain process
which will be used throughout the paper. The uncertain measure is a set function from a \( \sigma \)-algebra \( \mathcal{L} \)
genrated by a non-empty set \( \Gamma \) to the set of real numbers.

**Definition 1.** (Liu [8] and [10]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on nonempty set \( \Gamma \). A set function \( \mathcal{M} \) is called an
uncertain measure if it satisfies the following four axioms:

**Axiom 1.** (Normality) \( \mathcal{M}\{\Gamma\} = 1 \) for the universal set \( \Gamma \);

**Axiom 2.** (Duality) \( \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1 \) for any event \( \Lambda \);
Axiom 3. (Subadditivity) For every countable sequence of events \( \{ \Lambda_i \} \), we have

\[
\mathcal{M}\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.
\]

Axiom 4. (Product Axiom) Let \((\Gamma_i,\mathcal{L}_i,\mathcal{M}_i)\) be uncertainty spaces for \(i = 1, 2, \ldots\). Then the product uncertain measure \(\mathcal{M}\) is an uncertain measure satisfying

\[
\mathcal{M}\left( \prod_{i=1}^{\infty} \Lambda_i \right) = \bigwedge_{i=1}^{\infty} \mathcal{M}_i\{\Lambda_i\}.
\]

where \(\Lambda_i\) are arbitrarily chosen events from \(\mathcal{L}_i\) for \(i = 1, 2, \ldots\), respectively.

Definition 2. (Liu [8]) An uncertain variable is a measurable function \(\xi\) from an uncertainty space \((\Gamma,\mathcal{L},\mathcal{M})\) to the set of real numbers, i.e., for any Borel set \(B\) of real numbers, the set

\[
\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}
\]

is an event.

Definition 3. (Liu [10]) The uncertain variables \(\xi_1, \xi_2, \ldots, \xi_m\) are said to be independent if

\[
\mathcal{M}\left( \bigcap_{i=1}^{m} (\xi_i \in B_i) \right) = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}
\]

for any Borel sets \(B_1, B_2, \ldots, B_m\) of real numbers.

An uncertain variable is often described by an uncertainty distribution (Liu [8]). The uncertainty distribution \(\Phi\) of an uncertain variable \(\xi\) is defined by

\[
\Phi(x) = \mathcal{M}\{\xi \leq x\}
\]

for any real number \(x\). An uncertainty distribution \(\Phi\) is said to be regular (Liu [12]) if its inverse function \(\Phi^{-1}(\alpha)\) exists and is unique for each \(\alpha \in (0, 1)\).

Definition 4. (Liu [8]) The expected value of uncertain variable \(\xi\) is defined by

\[
E[\xi] = \int_{-\infty}^{\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr
\]

provided that at least one of the two integrals is finite.

It has been proved by Liu [12] that the expected value of \(\xi\) can also be written as

\[
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.
\]

Next, we will give several uncertainty distributions and their inverse uncertainty distributions, respectively.
Example 1. An uncertain variable $\xi$ is called linear if it has a linear uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x-a)/(b-a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b 
\end{cases}
$$

denoted by $L(a,b)$ where $a$ and $b$ are real numbers with $a < b$. The inverse uncertainty distribution of linear uncertain variable $L(a,b)$ is

$$
\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b.
$$

Example 2. (Liu [8]) An uncertain variable $\xi$ is called zigzag if it has a zigzag uncertainty distribution

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a; \\
(x-a)/2(b-a), & \text{if } a \leq x \leq b; \\
(x+c-2b)/2(c-b), & \text{if } b \leq x \leq c; \\
1, & \text{if } x \geq c 
\end{cases}
$$

denoted by $Z(a,b,c)$ where $a, b, c$ are real numbers with $a < b < c$. The inverse uncertainty distribution of zigzag uncertain variable $Z(a,b,c)$ is

$$
\Phi^{-1}(\alpha) = \begin{cases} 
(1-2\alpha)a + 2\alpha b, & \text{if } \alpha \leq 0.5; \\
(2-2\alpha)b + (2\alpha - 1), & \text{if } \alpha > 0.5. 
\end{cases}
$$

Example 3. An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution

$$
\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}
$$

denoted by $N(e,\sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$. The inverse uncertainty distribution of normal uncertain variable $N(e,\sigma)$ is

$$
\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1-\alpha}.
$$

Example 4. An uncertain variable $\xi$ is called lognormal if $\ln \xi$ is a normal uncertain variable $N(e,\sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution

$$
\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-\ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}
$$

denoted by $\log N(e,\sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$. The inverse uncertainty distribution of lognormal uncertain variable $\log N(e,\sigma)$ is

$$
\Phi^{-1}(\alpha) = \exp(e) \left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma/\pi}.
$$
The distribution of a monotonous function of uncertain variables can be easily obtained by the following theorem.

**Theorem 1. (Liu [10])** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent uncertain variables with uncertainty distributions \( \Phi_1, \Phi_2, \cdots, \Phi_n \), respectively. If \( f(x_1, x_2, \cdots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_2, \cdots, x_n \), then

\[
\xi = f(\xi_1, \xi_2, \cdots, \xi_n)
\]

is an uncertain variable with uncertainty distribution

\[
\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left( \min_{1 \leq i \leq m} \Phi_i(x_i) \land \left( \min_{m+1 \leq i \leq n} (1 - \Phi_i(x_i)) \right) \right)
\]

and inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n(1-\alpha)).
\]

Suppose that \( \xi_1, \xi_2, \cdots, \xi_n \) are independent and identical distributioned uncertain variables with uncertainty distribution \( \Phi \), then \( \xi = \xi_1 + \xi_2 + \cdots + \xi_n \) have an uncertainty distribution

\[
\Psi(x) = \Phi \left( \frac{x}{n} \right).
\]

Similar to random variables, convergence in distribution of uncertain variables is defined by the following definition.

**Definition 5. (Liu [8])** Let \( \xi, \xi_1, \xi_2, \cdots \) be a sequence of uncertain variables with uncertainty distributions \( \Phi, \Phi_1, \Phi_2, \cdots \), respectively, then \( \xi_i \) is said to converge in distribution to \( \xi \) if

\[
\lim_{i \to \infty} \Phi_i(x) = \Phi(x)
\]

at every continuous point of \( \Phi(x) \).

**Theorem 2. (Polyrectangular Theorem, Liu [10])** Let \( (\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \) and \( (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2) \) be two uncertainty spaces. Then the polyrectangle

\[
A = \bigcup_{i=1}^{m} (\Lambda_1^i \times \Lambda_2^i)
\]

on the product uncertainty space \( (\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \) and \( (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2) \) has an uncertain measure

\[
\mathcal{M}\{A\} = \bigvee_{i=1}^{m} \mathcal{M}_1\{\Lambda_1^i\} \land \mathcal{M}_2\{\Lambda_2^i\}.
\]
III. UNCERTAIN RENEWAL PROCESS

Uncertain Renewal Reward process

In this section, we shall recall some concepts about uncertain renewal reward process proposed by Liu [12].

Definition 6. (Liu [9]) Let $T$ be a totally index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any $t \in T$ and any Borel set $B$ of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$$

is an event.

Definition 7. (Liu [9]) Let $\xi_1, \xi_2, \cdots$ be independent and identically distributed (iid) positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N_t = \max_{n \geq 0} \{n | S_n \leq t\}$$

is called an uncertain renewal process.

Note that event $\{N_t \leq k\}$ is same with event $\{S_{k+1} \leq t\}$. For an uncertain renewal process, Liu [12] proved that $\frac{N_t}{t}$ converges in mean to $\frac{1}{\xi_1}$, i.e.,

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right].$$

Definition 8. (Liu [12]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be uncertain rewards. It is also assumed that $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ are independent. Then

$$R_t = \sum_{i=1}^{N_t} \eta_i$$

is called an uncertain renewal reward process, where $N_t$ is the uncertain renewal process.

Theorem 3. (Liu [12]) Let $R_t$ be an uncertain renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$. If those interarrival times $\xi_1, \xi_2, \xi_3, \cdots$ and rewards $\eta_1, \eta_2, \eta_3, \cdots$ have uncertainty distributions $\Phi$ and $\Psi$, then $R_t$ has an uncertainty distribution

$$\Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).$$

Here we set $x/k = \infty$ and $\Phi(x/k) = 1$ when $k = 0$.

Theorem 4. (Elementary uncertain renewal reward theorem, Liu [12]) Let $R_t$ be a renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$. If $E[\eta_1/\xi_1]$ exists, then

$$\lim_{t \to \infty} \frac{E[R_t]}{t} = E\left[\frac{\eta_1}{\xi_1}\right].$$
If those interarrival times $\xi_1, \xi_2, \cdots$ and rewards $\eta_1, \eta_2, \cdots$ have regular uncertainty distribution $\Phi$ and $\Psi$, then
\[
\lim_{t \to \infty} \frac{E[R_t]}{t} = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1 - \alpha)} d\alpha.
\]

Uncertain delayed renewal process

In the section we recall the so called uncertain delayed renewal process introduced by Zhang et al.\[21\]

**Definition 9.** (Uncertain delayed renewal process, Zhang et al.\[21\]) Let $\xi_1, \xi_2, \cdots$ be a sequence of independent positive uncertain variables. Assume that $\xi_1$ has a different uncertainty distribution from $\xi_2, \xi_3, \cdots$ which are identically distributed. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process
\[
N_t = \max_{n \geq 0} \{n | S_n \leq t\}
\]
is called an uncertain delayed renewal process.

**Remark 1.** An uncertain delayed renewal process $N_t$ degenerates to an uncertain renewal process if $\xi_1$ has the same uncertainty distribution as $\xi_2, \xi_3, \cdots$.

**Theorem 5.** (Zhang et al.\[21\]) Let $N_t$ be an uncertain delayed renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$. If $\xi_1$ has an uncertainty distribution $\Phi_1$ and $\xi_2, \xi_3, \cdots$ have a common uncertainty distribution $\Phi$, then $N_t$ has an uncertainty distribution
\[
\Psi_t(x) = 1 - \sup_{0 \leq s \leq t} \Phi_1(s) \wedge \Phi \left( \frac{t - s}{[x]} \right).
\]
Here $[x]$ represents the maximal integer less than or equal to $x$. Here, we set $(t - s)/[x] = \infty$ and $\Phi \left( (t - s)/[x] \right) = 1$ when $[x] = 0$.

**Remark 2.** When $\xi_1$ has the same uncertainty distribution as $\xi_2$, i.e., $\Phi_1(x) = \Phi(x)$, for any $x \in R$, we have
\[
\Phi_{[x]+1}(t) = \Phi \left( \frac{t}{[x] + 1} \right).
\]
In this case, $N_t$ has an uncertainty distribution
\[
\Psi_t(x) = 1 - \Phi \left( \frac{t}{[x] + 1} \right).
\]

**Theorem 6.** (Zhang et al.\[21\]) Let $N_t$ be an uncertain delayed renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$. Then
\[
\frac{N_t}{t} \to \frac{1}{\xi_2}
\]
in the sense of convergence in distribution.
Theorem 7. (Elementary uncertain delayed renewal theorem, Zhang et al. [21]) Let $N_t$ be an uncertain delayed renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ If $E\left[\frac{1}{\xi_2}\right]$ exists, then
\[
\lim_{t \to \infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_2}\right].
\]

IV. UNCERTAIN DELAYED RENEWAL REWARD PROCESS

In the section, we will introduce uncertain delayed renewal reward process and an elementary uncertain delayed renewal reward theorem. The elementary uncertain renewal reward theorem given by Liu in book [12] and elementary uncertain delayed renewal theorem given by Zhang et al. [21] are special cases of the elementary uncertain delayed renewal reward theorem which will be discussed in this section.

Definition 10. Let $\xi_1, \xi_2, \cdots$ be uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be uncertain rewards. It is also assumed that $\xi_1$ has a different uncertainty distribution from $\xi_2, \xi_3, \cdots$ which are identically distributed, and $\eta_1$ has a different uncertainty distribution from $\eta_2, \eta_3, \cdots$ which are identically distributed. It is also assumed that $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ are independent. Then
\[
R_t = \sum_{i=1}^{N_t} \eta_i
\]
is called an uncertain delayed renewal reward process, where $N_t$ is the delayed renewal process.

Remark 3. If $\xi_1$ has the same uncertainty distribution as $\xi_2, \xi_3, \cdots$ and $\eta_1$ has the same uncertainty distribution as $\eta_2, \eta_3, \cdots$, then $R_t$ is a renewal reward process.

Theorem 8. Let $R_t$ be an uncertain delayed renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$. If $\xi_1$ has an uncertainty distribution $\Phi_1$ and $\xi_2, \xi_3 \cdots$ have a common uncertainty distribution $\Phi$, $\eta_1$ has an uncertainty distribution $\Psi_1$, and $\eta_2, \eta_3, \cdots$ have a common uncertainty distribution $\Psi$. Then $R_t$ has an uncertainty distribution
\[
\Upsilon_t(x) = (1 - \Phi_1(t)) \lor \max_{k \geq 1} \left(1 - \sup_{0 \leq u \leq t} \Phi_1(u) \land \Phi\left(\frac{t - u}{k}\right)\right) \land \left(\sup_{0 \leq v \leq x} \Psi_1(v) \land \Psi\left(\frac{x - v}{k - 1}\right)\right).
\]
Here we set $(x - v)/(k - 1) = \infty$ and $\Phi((x - v)/(k - 1)) = 1$ when $k = 1$.

Proof. Note that $\xi_1, \xi_2, \xi_3, \cdots$ are independent, from the definition of delayed renewal reward process,
The uncertainty distribution of the delayed renewal process that

\[ \Upsilon_t(x) = \mathcal{M}\left\{ \sum_{i=1}^{N_t} \eta_i \leq x \right\} \]

\[ = \mathcal{M}\left\{ \bigcup_{k=0}^{\infty} (N_t = k) \bigcap \left( \sum_{i=1}^{k} \eta_i \leq x \right) \right\} \]

\[ = (1 - \Phi_1(t)) \lor \max_{k \geq 1} \mathcal{M}\{ N_t \leq k \} \land \mathcal{M}\left\{ \sum_{i=1}^{k} \eta_i \leq x \right\} \]

\[ = (1 - \Phi_1(t)) \lor \max_{k \geq 1} \mathcal{M}\{ N_t \leq k \} \land \mathcal{M}\left\{ \bigcup_{0 \leq v \leq x} (\eta_1 \leq v) \bigcap \left( \sum_{i=2}^{k} \eta_i \leq x - v \right) \right\} \]

\[ = (1 - \Phi_1(t)) \lor \max_{k \geq 1} \mathcal{M}\{ N_t \leq k \} \land \mathcal{M}\left\{ \bigcup_{0 \leq v \leq x} (\eta_1 \leq v) \bigcap \left( \sum_{i=2}^{k} \eta_i \leq x - v \right) \right\} \]

\[ = (1 - \Phi_1(t)) \lor \max_{k \geq 1} \left( 1 - \sup_{0 \leq u \leq t} \Phi_1(u) \land \Phi\left( \frac{t - u}{k} \right) \right) \land \left( \sup_{0 \leq v \leq x} \Psi_1(v) \land \Psi\left( \frac{x - v}{k - 1} \right) \right) \]

The theorem is proved.

**Corollary 1.** If the uncertainty distributions \( \Phi, \Psi, \Phi_1 \) and \( \Psi \) are regular, then \( R_t \) has an uncertainty distribution

\[ \Upsilon_t(x) = \sup_{k \geq 0}(1 - \Phi_{k+1}(t)) \land \Psi_k(x) \]

where \( \Phi_{k+1} \) and \( \Psi_k \) are nonnegative function such that \( \Phi_{k+1}^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + k\Phi^{-1}(\alpha) \) and \( \Psi_k^{-1}(\alpha) = \Psi_1^{-1}(\alpha) + (k - 1)\Psi^{-1}(\alpha) \), \( \Psi_0(x) = 1 \) for any integer \( k \geq 1 \).

**Proof:** If the uncertainty distributions \( \Phi, \Phi_1, \Psi_1 \) and \( \Psi \) are regular, then it follows from the operational law of uncertain variables that \( \sum_{i=1}^{k+1} \xi_i \) and \( \sum_{i=1}^{k} \eta_i \) have uncertainty distributions \( \Phi_{k+1} \) and \( \Psi_k \) (for any integer \( k \geq 1 \)) such that

\[ \Phi_{k+1}^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + k\Phi^{-1}(\alpha) \]

and

\[ \Psi_k^{-1}(\alpha) = \Psi_1^{-1}(\alpha) + (k - 1)\Psi^{-1}(\alpha). \]

Hence,

\[ \Upsilon_t(x) = (1 - \Phi_1(t)) \lor \sup_{k \geq 1} \mathcal{M}\{ N_t \leq k \} \land \mathcal{M}\left\{ \sum_{i=1}^{k} \eta_i \leq x \right\} \]

\[ = (1 - \Phi_1(t)) \lor \sup_{k \geq 1} \left( 1 - \Phi_{k+1}(t) \right) \land \Psi_k(x) \]

\[ = \sup_{k \geq 0} \left( 1 - \Phi_{k+1}(t) \right) \land \Psi_k(x). \]

**Remark 4.** If \( \Phi_1(x) = \Phi(x) \) and \( \Psi_1(x) = \Psi(x) \) for any nonnegative real number \( x \), then

\[ \Phi_{k+1}(x) = \Phi\left( \frac{x}{k + 1} \right) \]
and
\[ \Psi_k(x) = \Psi\left(\frac{x}{k}\right) \]
are true. What is more, the delayed renewal reward process \( R_t \) has an uncertainty distribution
\[ \Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right). \]

**Theorem 9.** Assume that \( R_t \) is a renewal reward process with uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and uncertain rewards \( \eta_1, \eta_2, \cdots \) Then the reward rate
\[ \frac{R_t}{t} \rightarrow \frac{\xi_2}{\eta_2} \]
in the sense of convergence in distribution as \( t \to \infty \).

**Proof:** It follows from Theorem 8 that the uncertainty distribution of \( R_t \) is
\[ \Upsilon_t(x) = (1 - \Phi_1(t)) \vee \max_{k \geq 1} \left(1 - \sup_{0 \leq u \leq t} \Phi_1(u) \wedge \Phi\left(\frac{t-u}{k}\right)\right) \wedge \left(\sup_{0 \leq v \leq t} \Psi_1(v) \wedge \Psi\left(\frac{x-v}{k-1}\right)\right) \]
where \( \Phi_1 \) and \( \Psi_1 \) are the uncertainty distributions of \( \xi_1 \) and \( \eta_1 \), respectively. Then \( R_t/t \) has an uncertainty distribution
\[ \Upsilon_t(x) = (1 - \Phi_1(t)) \vee \max_{k \geq 1} \left(1 - \sup_{0 \leq u \leq t} \Phi_1(u) \wedge \Phi\left(\frac{t-u}{k}\right)\right) \wedge \left(\sup_{0 \leq v \leq tx} \Psi_1(v) \wedge \Psi\left(\frac{tx-v}{k-1}\right)\right). \]
When \( t \to \infty \), we have
\[ \Upsilon_t(x) \rightarrow \sup_{y \geq 0} (1 - \Phi(y)) \wedge \Psi(xy) \]
which is just the uncertainty distribution of \( \eta_2/\xi_2 \).

**Theorem 10.** Let \( R_t \) be a delayed renewal reward process with uncertain interarrival times \( \xi_1, \xi_2, \cdots \) and uncertain rewards \( \eta_1, \eta_2, \cdots \) If \( E[\eta_2/\xi_2] \) exists, then
\[ \lim_{t \to \infty} \frac{E[R_t]}{t} = E\left[\frac{\eta_2}{\xi_2}\right]. \quad (1) \]
If those interarrival times \( \xi_2, \xi_3, \cdots \) and rewards \( \eta_2, \eta_3, \cdots \) have regular uncertainty distributions \( \Phi \) and \( \Psi \), then
\[ \lim_{t \to \infty} \frac{E[R_t]}{t} = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)} d\alpha. \quad (2) \]

**Proof.** It follows from Theorem 8 that the uncertainty distribution of \( R_t/t \) is
\[ \Upsilon_t(x) = (1 - \Phi_1(t)) \vee \max_{k \geq 1} \left(1 - \sup_{0 \leq u \leq t} \Phi_1(u) \wedge \Phi\left(\frac{t-u}{k}\right)\right) \wedge \left(\sup_{0 \leq v \leq tx} \Psi_1(v) \wedge \Psi\left(\frac{tx-v}{k-1}\right)\right) \]
and \( \xi_2/\eta_2 \) has an uncertainty distribution
\[ \Upsilon(x) = \sup_{y \geq 0} (1 - \Phi(y)) \wedge \Psi(xy). \]
Note that $\Upsilon_t(x) \to \Upsilon(x)$ and $\Upsilon_t(x) \geq \Upsilon(x)$ for all large enough $t$. It follows from Lebesgue dominated convergence theorem and the existence of $E[\eta_2/\xi_2]$ that

$$\lim_{t \to \infty} \frac{E[R_t]}{t} = \lim_{t \to \infty} \int_0^\infty (1 - \Upsilon_t(x))dx = \int_0^\infty (1 - \Upsilon(x))dx = E\left[\frac{\xi_2}{\eta_2}\right].$$

In addition, since the inverse uncertainty distribution of $\eta_2/\xi_2$ is just $\Psi^{-1}(\alpha)/\Phi^{-1}(1 - \alpha)$. The theorem is proved.

V. Applications

In this section, we will give some examples as applications for uncertain delayed renewal reward process theorems.

**Example 5.** Let $R_t$ be a delayed renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$. If those interarrival times $\xi_1, \xi_2, \xi_3, \cdots$ and rewards $\eta_1, \eta_2, \cdots$ have uncertainty distributions $\mathcal{L}(a_1, b_1), \mathcal{L}(a, b)$ and $\mathcal{L}(c_1, d_1), \mathcal{L}(c, d)$, then $R_t$ has an uncertainty distribution

$$\Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi_{k+1}(t)\right) \wedge \Psi_k(x)$$

where

$$\Phi_{k+1}(t) = \begin{cases} 
0, & \text{if } t \leq a_1 + ka \\
\frac{t - a_1 - ka}{b_1 + kb - a_1 - ka}, & \text{if } a_1 + ka \leq t \leq b_1 + kb \\
1, & \text{if } t \geq b_1 + kb
\end{cases}$$

and

$$\Psi_k(t) = \begin{cases} 
0, & \text{if } x \leq c_1 + kc \\
\frac{t - c_1 - kc}{d_1 + kd - c_1 - kd}, & \text{if } c_1 + kc \leq t \leq d_1 + kd \\
1, & \text{if } t \geq d_1 + kd
\end{cases}$$

It follows from the delayed renewal reward theorem that

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^1 \frac{(1 - \alpha)c + \alpha d}{\alpha a + (1 - \alpha)b}d\alpha.$$

**Example 6.** Let $R_t$ be a delayed renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$. If those interarrival times $\xi_1, \xi_2, \xi_3, \cdots$ and rewards $\eta_1, \eta_2, \cdots$ have uncertainty distributions $\mathcal{Z}(a_1, b_1, c_1), \mathcal{Z}(a, b, c)$ and $\mathcal{Z}(d_1, e_1, f_1), \mathcal{Z}(d, e, f)$, respectively, then $R_t$ has an uncertainty distribution

$$\Upsilon_t(x) = \max_{k \geq 0} \left(1 - \Phi_{k+1}(t)\right) \wedge \Psi_k(x)$$
where

\[ \Phi_{k+1}(t) = \begin{cases} 
0, & \text{if } t \leq a_1 + ka \\
\frac{t-a_1-ka}{2(b_1+kb-a_1-ka)}, & \text{if } a_1 + ka \leq t \leq b_1 + kb \\
\frac{t+c_1+kc-2b_1-2kb}{2(c_1+kc-b_1-ba)}, & \text{if } b_1 + kb \leq t \leq c_1 + kc \\
1, & \text{if } t \geq c_1 + kc
\end{cases} \]

and

\[ \Psi_{k}(t) = \begin{cases} 
0, & \text{if } x \leq d_1 + (k-1)d \\
\frac{x-d_1-(k-1)d}{2(e_1+(k-1)e-d_1-(k-1)d)}, & \text{if } d_1 + (k-1)d \leq x \leq e_1 + (k-1)e \\
\frac{x+f_1+(k-1)f-2e_1-2(k-1)e}{2(f_1+(k-1)f-e_1-(k-1)e)}, & \text{if } e_1 + (k-1)e \leq x \leq f_1 + (k-1)f \\
1, & \text{if } x \geq f_1 + (k-1)f.
\end{cases} \]

It follows from the delayed renewal reward theorem that

\[ \lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^{1/2} \left( \frac{2a(e-d)+d}{2a(a-b)+2b-a} + \frac{2a(e-f)+f}{2a(c-b)+2b-c} \right) d\alpha. \]

**Example 7.** Let \( R_t \) be a delayed renewal process with uncertain interarrival times \( \xi_1, \xi_2, \ldots \) and uncertain rewards \( \eta_1, \eta_2, \ldots \) If those interarrival times \( \xi_1, \xi_2, \xi_3 \) and rewards \( \eta_1, \eta_2, \ldots \) have uncertainty distributions \( N(e_1, \sigma_1), N(e_2, \sigma_2) \) and \( N(e_3, \sigma_3), N(e_4, \sigma_4) \), then \( R_t \) has an uncertainty distribution

\[ \Upsilon_t(x) = \max_{k \geq 0} (1 - \Phi_{k+1}(t)) \land \Psi_{k}(x) \]

where

\[ \Phi_{k+1}(t) = \left( 1 + \exp \left( \frac{\pi(e_1+ke_2-t)}{\sqrt{3}(\sigma_1+\sigma_2)} \right) \right)^{-1} \]

and

\[ \Psi_{k}(x) = \left( 1 + \exp \left( \frac{\pi(e_3+(k-1)e_4-x)}{\sqrt{3}(\sigma_3+(k-1)e_4)} \right) \right)^{-1}. \]

It follows from the delayed renewal reward theorem that

\[ \lim_{t \to \infty} \frac{E[N_t]}{t} = \int_0^{1/2} \frac{\pi e_4 + \sqrt{3}\sigma_4 \ln \frac{\alpha}{1-\alpha}}{\pi e_2 + \sqrt{3}\sigma_2 \ln \frac{\alpha}{1-\alpha}} d\alpha. \]

**Example 8.** Let \( R_t \) be a delayed renewal process with uncertain interarrival times \( \xi_1, \xi_2, \ldots \) and uncertain rewards \( \eta_1, \eta_2, \ldots \) If those interarrival times \( \xi_1, \xi_2, \xi_3 \) and rewards \( \eta_1, \eta_2, \ldots \) have uncertainty distributions \( LOGN(e_1, \sigma_1), LOGN(e_2, \sigma_2) \) and \( LOGN(e_3, \sigma_3), LOGN(e_4, \sigma_4) \), then \( R_t \) has an uncertainty distribution

\[ \Upsilon_t(x) = \max_{k \geq 0} (1 - \Phi_{k+1}(t)) \land \Psi_{k}(x) \]

where

\[ \Phi_{k+1}(t) = \left( 1 + \exp \left( \frac{\pi(e_1+ke_2-ln t)}{\sqrt{3}(\sigma_1+\sigma_2)} \right) \right)^{-1} \]
and

\[ \Psi_k(x) = \left(1 + \exp\left(\frac{\pi(e_4+(k-1)e_4-\ln x)}{\sqrt{3}(\sigma_3+(k-1)\sigma_4)}\right)\right)^{-1}. \]

It follows from the delayed renewal reward theorem that

\[ \lim_{t \to \infty} \frac{E[N_t]}{t} = \exp(e_4 - e_2) \int_0^1 \left(\frac{\alpha}{1-\alpha}\right)^{\frac{2}{3}(\sigma_4-\sigma_2)} d\alpha. \]

VI. Conclusion

This paper can be regarded as a continuation of the uncertain renewal reward process and uncertain delayed renewal process. A theory of uncertain delayed renewal reward process was developed. Particularly, the elementary uncertain delayed renewal reward theorem was established. At last, several examples were given.

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References


