Some New Ranking Criteria in Data Envelopment Analysis under Uncertain Environment

Meilin Wen\textsuperscript{a,b,*}, Zhongfeng Qin\textsuperscript{c}, Rui Kang\textsuperscript{b}

\textsuperscript{a}Science and Technology on Reliability and Environmental Engineering Laboratory
\textsuperscript{b}School of Reliability and Systems Engineering, Beihang University
\textsuperscript{c}School of Economics and Management Science, Beihang University

Beijing, 100191, China

wenmeilin@buaa.edu.cn, qin@buaa.edu.cn, kangruix@buaa.edu.cn

Abstract

Data Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision-making units (DMUs), which has been applied extensively to education, hospital, finance, etc. However, in real-world situations, the data of production processes can not be precisely measured in some cases, which leads to the research of DEA in uncertain environments. This paper will give some researches to uncertain DEA based on uncertainty theory. Due to the uncertain inputs and outputs, we will give three uncertain DEA models, as well as three types of fully ranking Criteria. For each uncertain DEA model, its crisp equivalent model is presented to simplify the computation of uncertain models. Finally, a numerical example is presented to illustrate the three ranking criteria.

Keywords: Data Envelopment Analysis; Ranking Criterion; Uncertainty theory; Uncertain variable; Uncertain measure

1 Introduction

Data envelopment analysis (DEA), as an useful management and decision tool, has been widely used since it was first invented by Charnes et al. [4]. This is followed by a series of theoretical extensions. See Banker et al. [2], Charnes et al. [5], Petersen [30], Tone [33] and Cooper [9]. More DEA papers can refer to Seiford [31] in which 500 references are documented.

In many cases, decision makers are interested in a complete ranking beyond the dichotomized classification. The researches on ranking have come up. By now many papers on ranking have been published over the last decade in DEA filed. By evaluating DMUs through both self and peer pressure, Sexton et al. [32] can attain a more balanced view of the decision-making units. Andersen & Petersen [1] developed the super-efficiency approach to get a ranking value which may be greater than one through evaluated
DMU’s exclusion from the linear constraints. In the benchmarking ranking method [34], a DMU is highly ranked if it is chosen as a useful target for many other DMUs.

Most methods of ranking DMUs assume that all inputs and outputs data are exactly known. However, in many situations, such as in a manufacturing system, a production process or a service system, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. Instead the data for evaluation are often collected from investigation to decide the natural language such as good, medium and bad rather than a specific case. That is the inputs and outputs are fuzzy. Cooper et al. [7] [8] were the first, to the best of our knowledge, to study how to deal with imprecise data in DEA. Other studies on fuzzy DEA include Kao & Liu [14], Entani et al. [10], Guo & Tanaka [13] and Lertworasirikul et al. [15]).

A lot of surveys showed that human uncertainty does not behave like fuzziness. For example, we say “the input is about 10”. Generally, we employ fuzzy variable to describe the concept of “about 10”, then there exists a membership function, such as a triangular one (9, 10, 11). Based on this membership function, we can obtain that the lifetime is “exactly 10” with possibility measure 1. On the other hand, the opposite event of “not exactly 10” has the same possibility measure. The conclusion that “not 10” and “exactly 10” have the same possibility measure is not appropriate. In order to have a better mathematical tool to deal with empirical data, uncertainty theory was founded by Liu [16] in 2007 and refined in 2010 [20]. As extensions of uncertainty theory, uncertain process and uncertain differential equations (Liu [17]), uncertain calculus (Liu [18]) were proposed. Uncertain programming was first proposed by Liu [19] in 2009, which wants to deal with the optimal problems involving uncertain variable. This work was followed by an uncertain multiobjective programming and an uncertain goal programming (Liu and Chen [22]), and an uncertain multilevel programming (Liu & Yao [23]). Since that, uncertainty theory was used to solve variety of real optimal problems, including finance (Chen & Liu [6], Peng [29], Liu [24]), reliability analysis (Liu [21], Zeng et al. [37]), graph (Gao [11], Gao & Gao [12]), et al.

This paper is organized as follows: Some basic concept and results on uncertainty theory will be introduced in Section 2; In Section 3, we will give some basic introduction to DEA models; In Section 4-6, we will give three uncertain DEA models, three fully ranking Criteria, as well as their equivalent deterministic models. Finally, a numerical example will be given to illustrate the uncertain DEA model and the ranking method in Section 7.

2 Preliminaries

Uncertainty theory was founded by Liu [16] in 2007 and refined by Liu [20] in 2010. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the
remainder of this paper.

Let \( \Gamma \) be a nonempty set, and \( L \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \in L \) is assigned a number \( M(\Lambda) \in [0, 1] \). In order to ensure that the number \( M(\Lambda) \) has certain mathematical properties, Liu [16][20] presented the three axioms:

(i) \( M(\Gamma) = 1 \) for the universal set \( \Gamma \).

(ii) \( M(\Lambda) + M(\Lambda^c) = 1 \) for any event \( \Lambda \).

(iii) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i)
\]

The triplet \((\Gamma, L, M)\) is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [18], thus producing the fourth axiom of uncertainty theory:

(iv) Let \((\Gamma_k, L_k, M_k)\) be uncertainty spaces for \( k = 1, 2, \ldots, \infty \). Then the product uncertain measure \( M \) is an uncertain measure satisfying

\[
M\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} M_k(\Lambda_k).
\]

An uncertain variable is a measurable function \( \xi \) from an uncertainty space \((\Gamma, L, M)\) to the set of real numbers (Liu [16]). In order to describe an uncertain variable in practice, the concept of uncertainty distribution is defined as

\[
\Phi(x) = M\{\xi \leq x\}
\]

for any real number \( x \). For example, the linear uncertain variable \( \xi \sim \mathcal{L}(a, b) \) has an uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/(b - a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b.
\end{cases}
\]

An uncertain variable \( \xi \) is called zigzag if it has a zigzag uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/2(b - a), & \text{if } a \leq x \leq b \\
(x + c - 2b)/2(c - b), & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c
\end{cases}
\]

denoted by \( \mathcal{Z}(a, b, c) \) where \( a, b, c \) are real numbers with \( a < b < c \).
An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

(4)
denoted by $\mathcal{N}(e,\sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$.

An uncertainty distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$.

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\}$$

(5)

for any Borel sets $B_1, B_2, \ldots, B_n$.

**Theorem 1** (Liu [20]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

(6)
is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1} = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

(7)

**Theorem 2** (Liu and Ha [25]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

$$E[\xi] = \int_{0}^{1} f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) d\alpha$$

(8)

provided that $E[\xi]$ exists.

3 DEA Model

CCR model is one of the most frequently used DEA model, which was proposed by Charnes et al. [4]. Since the following sections will use this model, we will give some basic introduction to CCR model.

Firstly let us review some symbols and variables:

- DMU$_k$: the $k$th DMU, $k = 1, 2, \ldots, n$;
- DMU$_0$: the target DMU;
- $x_k \in R^{p \times 1}$: the inputs vector of DMU$_k$, $k = 1, 2, \ldots, n$;
- $x_0 \in R^{p \times 1}$: the inputs vector of the target DMU$_0$;
- $y_k \in R^{q \times 1}$: the outputs vector of DMU$_i$, $k = 1, 2, \ldots, n$;
\( y_0 \in \mathbb{R}^{q \times 1} \): the outputs vector of the target DMU;  
\( u \in \mathbb{R}^{p \times 1} \): the vector of input weights;  
\( v \in \mathbb{R}^{q \times 1} \): the vector of output weights.

In this model, the efficiency of entity evaluated is obtained as a ratio of the weighted output to the weighted input subject to the condition that the ratio for every entity is not larger than 1. Mathematically, it is described as follows:

\[
\max_{\theta} \frac{v^T y_0}{u^T x_0} 
\quad \text{subject to:} 
\]

\[
v^T y_j \leq u^T x_j, \ j = 1, 2, \ldots, n
\]

\[
u \geq 0 \quad v \geq 0.
\]

(9)

**Definition 1** (Efficiency) DMU \( \theta^* = 1 \), where \( \theta^* \) is the optimal value of (9).

4 Uncertain DEA Ranking Criteria

In many situations, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. This inspired many researchers to apply probability to DEA. As we know, probability or statistics needs a large amount of historical data. In the vast majority of real cases, the sample size is too small (even no-sample) to estimate a probability distribution. Then we have to invite some domain experts to evaluate their degree of belief that each event will occur. This section will give some researches to empirical uncertain DEA using the theory introduced in Section 2. The new symbols and notations are given as follows:

\( \tilde{x}_k = (\tilde{x}_{k1}, \tilde{x}_{k2}, \ldots, \tilde{x}_{kp}) \): the fuzzy input vectors of DMU, \( k = 1, 2, \ldots, n \);

\( \tilde{y}_k = (\tilde{y}_{k1}, \tilde{y}_{k2}, \ldots, \tilde{y}_{kq}) \): the fuzzy output vectors of DMU, \( k = 1, 2, \ldots, n \);

\( \Phi_k(x) = (\Phi_{k1}(x), \Phi_{k2}(x), \ldots, \Phi_{kp}(x)) \): the uncertainty distribution vector of \( \tilde{x}_k = (\tilde{x}_{k1}, \tilde{x}_{k2}, \ldots, \tilde{x}_{kp}) \), \( k = 1, 2, \ldots, n \);

\( \Psi_k(x) = (\Psi_{k1}(x), \Psi_{k2}(x), \ldots, \Psi_{kq}(x)) \): the uncertainty distribution vector of \( \tilde{y}_k = (\tilde{y}_{k1}, \tilde{y}_{k2}, \ldots, \tilde{y}_{kq}) \), \( k = 1, 2, \ldots, n \).

In the following sections, three types of uncertain DEA fully ranking criteria are to be investigated.

4.1 The Expected Ranking Criterion

Liu [16][18] proposed the expected value operator of uncertain variable and uncertain expected value model. The essential idea of the uncertain expected DEA model is to optimize the expected value of \( \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \) subject to some chance constraints, then we have the first type of the uncertain DEA model:
\[
\begin{align*}
\theta &= \max_{u, v} \mathbb{E} \left[ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \right] \\
\text{subject to :} & \\
M \{ v^T \tilde{y}_k \leq u^T \tilde{x}_k \} & \geq \alpha, \ k = 1, 2, \cdots, n \\
u & \geq 0 \\
v & \geq 0
\end{align*}
\]

(10)

in which \( \alpha \in (0.5, 1] \).

**Definition 2** A vector \((u, v) \geq 0\) is called a feasible solution to the uncertain programming model (10) if

\[
M \{ v^T \tilde{y}_k \leq u^T \tilde{x}_k \} \geq \alpha
\]

(11)

for \( k = 1, 2, \cdots, n \).

**Definition 3** A feasible solution \((u^*, v^*)\) is called an expected optimal solution to the uncertain programming model (10) if

\[
\mathbb{E} \left[ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \right] \geq \mathbb{E} \left[ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \right]
\]

(12)

for any feasible solution \((u, v)\).

**Expected Ranking Criterion:** The greater the optimal objective value is, the more efficient DMU\(_0\) is ranked.

**Theorem 3** Assume that \(\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}\) are independent uncertain inputs with uncertainty distribution \(\Phi_{1i}, \Phi_{2i}, \cdots, \Phi_{ni}\) for each \(i, i = 1, 2, \cdots, p\), and \(\tilde{y}_{1i}, \tilde{y}_{2i}, \cdots, \tilde{y}_{ni}\) are independent uncertain outputs with uncertainty distribution \(\Psi_{1j}, \Psi_{2j}, \cdots, \Psi_{nj}\) for each \(j, j = 1, 2, \cdots, q\). Then the uncertain programming model (10) is equivalent to the following model:

\[
\begin{align*}
\theta &= \max_{u, v} \int_0^1 \frac{v^T \Psi_k^{-1}(\alpha)}{u^T \Phi_k^{-1}(1 - \alpha)} \, d\alpha. \\
\text{subject to :} & \\
v^T \Psi_k^{-1}(\alpha) & \leq u^T \Phi_k^{-1}(1 - \alpha), \ k = 1, 2, \cdots, n \\
u & \geq 0 \\
v & \geq 0
\end{align*}
\]

(13)

**Proof:** Since the function \(\frac{v^T \tilde{y}}{u^T \tilde{x}}\) is strictly increasing with respect to \(\tilde{y}\) and strictly decreasing with respect to \(\tilde{x}\), it follows from Theorem 1 that the inverse uncertainty distribution of \(\frac{v^T \tilde{y}}{u^T \tilde{x}}\) is \(\frac{v^T \Psi_k^{-1}(\alpha)}{u^T \Phi_k^{-1}(1 - \alpha)}\). Thus \(M \{ v^T \tilde{y}_k \leq u^T \tilde{x}_k \} \geq \alpha\) holds if and only if \(v^T \Psi_k^{-1}(\alpha) \leq u^T \Phi_k^{-1}(1 - \alpha)\) for \(k = 1, 2, \cdots, n\).

By using Theorem 2, we obtain

\[
\mathbb{E} \left[ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \right] = \int_0^1 \frac{v^T \Psi_0^{-1}(\alpha)}{u^T \Phi_0^{-1}(1 - \alpha)} \, d\alpha.
\]

(14)

The theorem is thus verified.
5 The Optimistic Ranking Criterion

Chance-constrained programming (CCP), which was initialized by Charnes & Cooper [3], offers a powerful means for modelling stochastic decision systems. The essential idea of chance-constrained programming is to optimize some critical value with a given confidence level subject to some chance constraints. Inspired by this idea, Liu [20] extended it to uncertain programming models. Assuming that the decision makers want to maximize the optimistic value of the uncertain objective at given confidence level, we have the second type of DEA model:

\[
\begin{align*}
\max_{\mathbf{u}, \mathbf{v}} & \quad \mathcal{F} \\
\text{subject to :} & \quad M \left\{ \frac{\mathbf{v}^T \mathbf{y}_0}{\mathbf{u}^T \mathbf{x}_0} \geq \mathcal{F} \right\} \geq 1 - \alpha \\
& \quad M \{ \mathbf{v}^T \mathbf{y}_k \leq \mathbf{u}^T \mathbf{x}_k \} \geq \alpha, \ k = 1, 2, \cdots, n \\
& \quad \mathbf{u} \geq 0 \\
& \quad \mathbf{v} \geq 0
\end{align*}
\]

(15)

in which \( \alpha \in (0.5, 1] \).

**Definition 4** A feasible solution \((\mathbf{u}^*, \mathbf{v}^*)\) is called an optimistic optimal solution to the uncertain programming model (15) if

\[
\max \left\{ \mathcal{F} \mid M \left\{ \frac{\mathbf{v}^T \mathbf{y}_0}{\mathbf{u}^T \mathbf{x}_0} \geq \mathcal{F} \right\} \geq 1 - \alpha \right\} \geq \max \left\{ \mathcal{F} \mid M \left\{ \frac{\mathbf{v}^T \mathbf{y}_0}{\mathbf{u}^T \mathbf{x}_0} \geq \mathcal{F} \right\} \geq 1 - \alpha \right\}
\]

(16)

for any feasible solution \((\mathbf{u}, \mathbf{v})\).

**Optimistic Ranking Criterion:** The greater the optimal objective value is, the more efficient DMU\(_0\) is ranked.

**Theorem 4** Assume that \(\tilde{x}_{1i}, \tilde{x}_{2i}, \cdots, \tilde{x}_{ni}\) are independent uncertain inputs with uncertainty distribution \(\Phi_{1i}, \Phi_{2i}, \cdots, \Phi_{ni}\) for each \(i, \ i = 1, 2, \cdots, p\), and \(\tilde{y}_{1i}, \tilde{y}_{2i}, \cdots, \tilde{y}_{ni}\) are independent uncertain outputs with uncertainty distribution \(\Psi_{1j}, \Psi_{2j}, \cdots, \Psi_{nj}\) for each \(j, \ j = 1, 2, \cdots, q\). Then the uncertain programming model (15) is equivalent to the following model:

\[
\begin{align*}
\max_{\mathbf{u}, \mathbf{v}} & \quad \mathbf{v}^T \Psi_0^{-1}(\alpha) \\
\text{subject to :} & \quad \mathbf{u}^T \Phi_0^{-1}(1 - \alpha) \\
& \quad \mathbf{u}^T \Psi_k^{-1}(\alpha) \leq \mathbf{u}^T \Phi_k^{-1}(1 - \alpha), \ k = 1, 2, \cdots, n \\
& \quad \mathbf{u} \geq 0 \\
& \quad \mathbf{v} \geq 0
\end{align*}
\]

(17)

**Proof:** By using Theorem 1, the theorem can be easily obtained.
6 The Maximal Chance Ranking Criterion

Sometimes the decision maker may want to maximize the chance of satisfying the event $\frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \geq 1$. In order to model this type of decision system, Liu [26] [27] [28] provided the dependent-chance programming (DCP). Here we carried out the DCP model into the DEA as follows:

$$\begin{aligned}
\theta &= \max_{u,v} \{ v^T \tilde{y}_0 \} \\
&\text{subject to :} \\
&\quad M \{ v^T \tilde{y}_k \leq u^T \tilde{x}_k \} \geq \alpha, \quad k = 1, 2, \ldots, n \\
&\quad u \geq 0 \\
&\quad v \geq 0 \\
\end{aligned}$$

in which $\alpha \in (0, 1]$.

**Definition 5** A feasible solution $(u^*, v^*)$ is called an maximal chance optimal solution to the uncertain programming model (18) if

$$M \{ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \geq 1 \} \geq M \{ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \geq \tilde{f} \}$$

for any feasible solution $(u, v)$.

**Maximal Chance Ranking Criterion**: The greater the optimal objective value is, the more efficient DMU$_0$ is ranked.

**Theorem 5** Assume that $\tilde{x}_{1i}, \tilde{x}_{2i}, \ldots, \tilde{x}_{ni}$ are independent uncertain inputs with uncertainty distribution $\Phi_{1i}, \Phi_{2i}, \ldots, \Phi_{ni}$ for each $i$, $i = 1, 2, \ldots, p$, and $\tilde{y}_{1i}, \tilde{y}_{2i}, \ldots, \tilde{y}_{mi}$ are independent uncertain outputs with uncertainty distribution $\Psi_{1j}, \Psi_{2j}, \ldots, \Psi_{nj}$ for each $j$, $j = 1, 2, \ldots, q$. Then the uncertain programming model (18) is equivalent to the following model:

$$\begin{aligned}
\theta &= \max_{u,v} \{ \frac{v^T \tilde{y}_0}{u^T \tilde{x}_0} \} \\
&\text{subject to :} \\
&\quad v^T \Phi_k^{-1}(\alpha) \leq u^T \Phi_k^{-1}(1 - \alpha), \quad k = 1, 2, \ldots, n \\
&\quad u \geq 0 \\
&\quad v \geq 0. \\
\end{aligned}$$

**Proof**: By using Theorem 1, the theorem can be easily obtained.

7 A Numerical Example

This example wants to illustrate the three uncertain DEA models and their corresponding ranking methods. For simplicity, we will only consider five DMUs with two inputs and two outputs which are all zigzag uncertain variables denoted by $Z(a, b, c)$. Table 1 gives the information of the DMUs.
Table 1: DMUs with two uncertain inputs and two uncertain outputs

<table>
<thead>
<tr>
<th>DMU</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input 1</td>
<td>$Z(3.5,4.0,4.5)$</td>
<td>$Z(2.9,2.9,2.9)$</td>
<td>$Z(4.4,4.9,5.4)$</td>
<td>$Z(3.4,4.1,4.8)$</td>
<td>$Z(5.9,6.5,7.1)$</td>
</tr>
<tr>
<td>Input 2</td>
<td>$Z(2.9,3.1,3.3)$</td>
<td>$Z(1.4,1.5,1.6)$</td>
<td>$Z(3.2,3.6,4.0)$</td>
<td>$Z(2.1,2.3,2.5)$</td>
<td>$Z(3.6,4.1,4.6)$</td>
</tr>
<tr>
<td>Output 1</td>
<td>$Z(2.4,2.6,2.8)$</td>
<td>$Z(2.2,2.2,2.2)$</td>
<td>$Z(2.7,3.2,3.7)$</td>
<td>$Z(2.5,2.9,3.3)$</td>
<td>$Z(4.4,5.1,5.8)$</td>
</tr>
<tr>
<td>Output 2</td>
<td>$Z(3.8,4.1,4.4)$</td>
<td>$Z(3.3,3.5,3.7)$</td>
<td>$Z(4.3,5.1,5.9)$</td>
<td>$Z(5.5,5.7,5.9)$</td>
<td>$Z(6.5,7.4,8.3)$</td>
</tr>
</tbody>
</table>

Table 2: Expected ranking results with different $\alpha$ confidence level

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>DMU 1</th>
<th>DMU 2</th>
<th>DMU 3</th>
<th>DMU 4</th>
<th>DMU 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.82</td>
<td>1</td>
<td>0.91</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.6</td>
<td>0.85</td>
<td>1</td>
<td>0.92</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.7</td>
<td>0.89</td>
<td>1</td>
<td>0.94</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.8</td>
<td>0.90</td>
<td>1</td>
<td>0.98</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.9</td>
<td>0.91</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

From Table 2, 3 and 4, we can get the following conclusions:

(i) Roughly speaking, the ranking results are DMU 2, DMU 4, DMU 5, DMU 3, DMU 1;

(ii) The confidence level $\alpha$ affects the ranking results. When $\alpha = 0.90$, the DMUs are ranked: DMU 3, DMU 4, DMU 5, DMU 2, DMU 1. At other $\alpha$, the DMUs are ranked: DMU 2, DMU 4, DMU 5, DMU 3, DMU 1; This phenomena indicates that the ranking method in uncertain environment is more complex than the traditional ranking methods because of the inherent uncertainty contained in inputs and outputs.

(iii) Although the ranking results with different ranking criterion are uniform in this example, the three ranking criterion are different in nature.

Table 3: Optimistic ranking results with different $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>DMU 1</th>
<th>DMU 2</th>
<th>DMU 3</th>
<th>DMU 4</th>
<th>DMU 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.82</td>
<td>1</td>
<td>0.89</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.6</td>
<td>0.85</td>
<td>1</td>
<td>0.91</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.7</td>
<td>0.89</td>
<td>1</td>
<td>0.94</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.8</td>
<td>0.90</td>
<td>1</td>
<td>0.98</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.9</td>
<td>0.91</td>
<td>0.99</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 4: Maximal ranking results with different $\alpha$

<table>
<thead>
<tr>
<th>confidence level $\alpha$</th>
<th>DMU$_1$</th>
<th>DMU$_2$</th>
<th>DMU$_3$</th>
<th>DMU$_4$</th>
<th>DMU$_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.10</td>
<td>0.50</td>
<td>0.31</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>0.60</td>
<td>0.06</td>
<td>0.40</td>
<td>0.26</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.70</td>
<td>0.03</td>
<td>0.30</td>
<td>0.22</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>0.80</td>
<td>0</td>
<td>0.20</td>
<td>0.19</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>0.90</td>
<td>0</td>
<td>0.08</td>
<td>0.12</td>
<td>0.10</td>
<td>0.10</td>
</tr>
</tbody>
</table>

8 Conclusion

Due to its widely practical used background, data envelopment analysis (DEA) has become a pop area of research. Since the data cannot be precisely measured in some practical cases, many paper have been published when the inputs and outputs are uncertain. This paper gave some researches to uncertain DEA based on uncertainty measure. Three uncertain DEA models have been proposed, which led to three fully ranking criteria. In order to simplify the computation of the uncertain DEA model, we have presented their equivalent crisp models. The numerical example illustrated the uncertain DEA models and the ranking methods.

Acknowledgments

This work was supported by National Natural Science Foundation of China (No.71201005).

References


