An uncertain optimal control model with Hurwicz criterion

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A new uncertain optimal control model based on the Hurwicz criterion is introduced to design dynamic optimization problems. Applying the method of dynamic programming, the principle of optimality is presented and then the equation of optimality is given to solve the proposed model. As an application, a manufacturing technology diffusion problem is discussed by using the equation obtained.

1. Introduction

Optimal control theory is one of the earliest and important branches of modern control theory, and the pioneer work can be traced to the end of the 1630s. One of the forceful methods to solve optimal control problems is based on dynamic programming. Dynamic programming, originated by Bellman in the 1950s, can be used to deal with various optimization problems. The basic idea of this method applied to optimal control is to consider a family of optimal control problems with different initial times and states, to establish relationships among these problems via the Hamilton–Jacobi–Bellman equation (HJB). This dynamic programming method and Pontryagin’s maximum principle are the two theoretical principles for optimal control theory. Nowadays, with the greater use of approaches and results in mathematics and computer science, optimal control theory has made an extraordinary development, and has been investigated and widely applied to many fields such as space technology, management, economics, and even medicine. For example, a multi-drug cancer chemotherapy model was introduced in [1] to simulate the possible response of the tumor cells under drug administration, and the model was formulated as an optimal control problem to minimize the tumor size under a set of constraints.

The real-world makes the control systems we face nondeterministic cases in numerous different forms. Among them, randomness is a rudimentary type of objective indeterminacy. A stochastic control system, which is described by a stochastic differential equation, was introduced to characterize a system with white noise. The focus on the investigation of stochastic optimal control initiated in 1960s, especially for finance, such as Merton [2]. A classical example was arisen in Merton’s model of optimal portfolio. Also the use of dynamic programming in optimization over Ito’s process was discussed by Dixit and Pindyck [3].

However, human uncertainty also brings inherent vagueness to the survey. It places many impacts on the search for optimal results. For modeling this indeterminacy, Liu [4] introduced an uncertain measure which can be interpreted to measure the personal belief degree of an uncertain event. Based on this fundamental concept, an uncertainty theory was founded [4] in 2007 and refined by Liu [5] in 2010 then became a branch of axiomatic mathematics for studying human uncertainty. Up to now, uncertainty theory has been developed in a variety of directions, including uncertain programming, uncertain statistics, uncertain risk analysis, and etc. Uncertain process and canonical process were introduced by Liu [6] as counterparts of...
stochastic process and Wiener process, respectively. Afterwards, the concept of uncertain differential equation driven by canonical process was presented in 2008 [6].


Inspired by our preceding work, a new uncertain optimal control model based on the Hurwicz criterion is formulated to study the uncertain control system characterized by an uncertain differential equation in an alternative approach. The remainder of this paper is organized as follows. Next section reviews some basic concepts in uncertainty theory and Hurwicz criterion. In Section 3, by using the Hurwicz criterion, a new optimal control model is established. Employing Bellman’s principle of optimality, a general solution procedure is obtained. Section 4 provides a type of manufacturing technology diffusion problem as an example to illustrate the application of the proposed uncertain optimal control model. The final section is the conclusion of this article.

2. Preliminary

2.1. Uncertainty theory

In this section, we will state some basic concepts in the uncertainty theory [4]. Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \lambda \in \mathcal{L} \) is called an event. An uncertain measure is a set function \( \mathcal{M} \) defined on the \( \sigma \)-algebra over \( \mathcal{L} \) if it satisfies three axioms: normality, self-duality, and countable subadditivity. Gao [16] studied some properties of an uncertain measure.

And the triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space. An uncertain variable is a measurable function from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers. The uncertain variables \( \zeta_1, \zeta_2, \ldots, \zeta_m \) are said to be independent if \( \mathcal{M}(\bigcap_{i=1}^{m} \{\zeta \in B_i\}) = \min_{i \in \{1, 2, \ldots, m\}} \mathcal{M}(\{\zeta \in B_i\}) \) for any Borel sets \( B_1, B_2, \ldots, B_m \) of real numbers. The distribution \( \Phi : R \to [0, 1] \) of an uncertain variable \( \zeta \) is defined by \( \Phi(x) = \mathcal{M}(\{\gamma \in \Gamma | \zeta(\gamma) \leq x\}) \) for \( x \in R \).

**Definition 2.1** (Liu [4]). Let \( \zeta \) be an uncertain variable, and \( \alpha \in (0, 1) \). Then \( \zeta_{\sup}(\alpha) = \sup\{r | \mathcal{M} \{ \zeta \geq r \} \geq \alpha \} \) is called the \( \alpha \)-optimistic value to \( \zeta \); and \( \zeta_{\inf}(\alpha) = \inf\{r | \mathcal{M} \{ \zeta \leq r \} \geq \alpha \} \) is called the \( \alpha \)-pessimistic value to \( \zeta \).

**Example 2.1** (Liu [5]). Let \( \zeta \) be a normal uncertain variable \( \mathcal{N}(\mu, \sigma) \). Then its \( \alpha \)-optimistic and \( \alpha \)-pessimistic values are

\[
\zeta_{\sup}(\alpha) = e - \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}, \quad \zeta_{\inf}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

**Theorem 2.1** (Liu [4,5]). Assume that \( \zeta \) is an uncertain variable. Then we have

(a) if \( \lambda \geq 0 \), then \( (\lambda \zeta)_{\sup}(\alpha) = \lambda \zeta_{\sup}(\alpha) \), and \( (\lambda \zeta)_{\inf}(\alpha) = \lambda \zeta_{\inf}(\alpha) \);

(b) if \( \lambda < 0 \), then \( (\lambda \zeta)_{\sup}(\alpha) = \lambda \zeta_{\inf}(\alpha) \), and \( (\lambda \zeta)_{\inf}(\alpha) = \lambda \zeta_{\sup}(\alpha) \);

(c) \( (\zeta + \eta)_{\sup}(\alpha) = \zeta_{\sup}(\alpha) + \eta_{\sup}(\alpha) \), \( (\zeta + \eta)_{\inf}(\alpha) = \zeta_{\inf}(\alpha) + \eta_{\inf}(\alpha) \) if \( \zeta \) and \( \eta \) are independent.

**Theorem 2.2** (Sheng and Zhu [15]). Let \( \zeta \) be a normal uncertain variable \( \mathcal{N}(\mu, \sigma) \). Then for any real number \( a \), and any small enough \( \varepsilon > 0 \),

\[
[a \zeta + b \zeta^2]_{\sup}(\alpha) \geq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} |a| \sigma + \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right)^2 b \sigma^2,
\]

\[
[a \zeta + b \zeta^2]_{\inf}(\alpha) \leq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \varepsilon}{\alpha - \varepsilon} |a| \sigma + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \varepsilon}{\varepsilon} \right)^2 b \sigma^2
\]

if \( b > 0 \); and

\[
[a \zeta + b \zeta^2]_{\sup}(\alpha) \geq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha - \varepsilon}{\alpha + \varepsilon} |a| \sigma + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \varepsilon}{\varepsilon} \right)^2 b \sigma^2,
\]

\[
[a \zeta + b \zeta^2]_{\inf}(\alpha) \leq \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} |a| \sigma + \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right)^2 b \sigma^2
\]
if $b < 0$; and also
\[
[a_z + b_z^2]_{\sup} (\alpha) = \frac{\sqrt{\pi}}{\pi} \ln \frac{1 - \alpha}{\alpha} |a| \sigma
\]
if $b = 0$.

**Definition 2.2 (Liu [6]).** An uncertain process $C_t$ is said to be a canonical process if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) $C_t$ has stationary and independent increments; (iii) every increment $C_{s,t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is
\[
\Phi(x) = \left(1 + \exp \left(\frac{-\pi x}{\sqrt{3} t}\right)\right)^{-1}, \quad x \in \mathbb{R}.
\]

Uncertain calculus is recommended to deal with differentiation and integration of uncertain processes.

**Definition 2.3 (Liu [7]).** Let $X_t$ be an uncertain process and let $C_t$ be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \ldots < t_k = b$, the mesh is written as
\[
\Delta = \max_{1 \leq i < k} |t_i - t_{i+1}|.
\]
Then the uncertain integral of $X_t$ with respect to $C_t$ is
\[
\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_t |_{t_i} | (C_{t_{i+1}} - C_{t_i})
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be integrable.

**Definition 2.4 (Liu [7]).** Let $C_t$ be a canonical process and let $Z_t$ be an uncertain process. If there exist two uncertain processes $\mu_t$ and $\sigma_t$ such that
\[
Z_t = Z_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dC_s
\]
for any $t \geq 0$, then we say $Z_t$ has an uncertain differential
\[
dZ_t = \mu_s dt + \sigma_s dC_t.
\]
In this case, $Z_t$ is called an uncertain process with drift $\mu_t$ and diffusion $\sigma_t$.

**Definition 2.5 (Liu [6]).** Suppose $C_t$ is a canonical process, $f$ and $g$ are some given functions. Then
\[
dx_t = f(t, X_t) dt + g(t, X_t) dC_t
\]
is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies (2.1) identically in $t$.

An existence and uniqueness theorem for the uncertain differential equation was proved by Chen and Liu [17] in 2010. Note that if $X_t$ is an uncertain vector, $f$ and $g$ are vector-value function and matrix-value function, respectively, $C_t$ is a multi-dimensional uncertain canonical process, then (2.1) is a multi-dimensional system of uncertain differential equations. And the existence and uniqueness theorem in [17] still holds for multi-dimensional uncertain differential equation.

### 2.2. Hurwicz criterion

Grounded on uncertain measure, the optimistic value criterion and pessimistic value criterion of uncertain variables have been introduced for handling optimization problems in uncertain environments. Applied the optimistic value criterion to consider the objectives is essentially a maximax approach, which maximizes the maximum uncertain return. This approach suggests that the decision maker who is attracted by high payoffs to take some adventure. As opposed to the optimistic value criterion, using the pessimistic value criterion for uncertain decision system is essentially a maximin approach, which the underlying philosophy is based on selecting the alternative that provides the least bad uncertain return. It suggests the decision maker who is in pursuit of cautious that there is at least a known minimum payoff in the event of an unfavorable outcome.

The Hurwicz criterion can also be called optimism coefficient method, designed by economics professor Leonid Hurwicz [18] in 1951. It is a complex decision making criterion attempting to find the intermediate area between the extremes posed by the optimistic and pessimistic criteria. Instead of assuming totally optimistic or pessimistic, Hurwicz criterion incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism. With the Hurwicz criterion, the decision maker first should subjectively select a coefficient $\rho$ denoting the optimism degree, note that
is an uncertain variable. Therefore we cannot regard the objective function as a real function to be optimized. We have to convert the uncertain objective to its crisp equivalent. In fact, there are many approaches to do so and they satisfy various requirements in different problems. These approaches are established due to some criteria including, for instance, expected value, optimistic value, pessimistic value, and etc. [19,20]. In [15], we consider making use of the optimistic value criterion. However, both the optimistic criterion and the pessimistic criterion are extreme criteria, in order to surmount the extreme cases, we may optimize the uncertain objectives applying a more flexible criterion.

3. Uncertain optimal control model with Hurwicz criterion

Considering a dynamic system, which is described by an uncertain differential equation, the controllers must select a best decision among all possible ones such that their objective function related to an uncertain process is optimized. Such optimization problems are called uncertain optimal control problems. An uncertain optimal control system guided by the following uncertain differential equation:

\[ dX_t = b(s, X_t, u_t)ds + \sigma(s, X_t, u_t)dc_t, \]

where \( X_t \) is the state variable, \( u_t \) is a control (represents the function \( u(t, x) \) of time \( t \) and state \( x \)), and \( c_t \) is a canonical process.

In uncertain environments, an important issue of the uncertain optimal control problem is: the meaning of optimal objective is not clear at all since the state \( x_t \) is an uncertain variable. Therefore we cannot regard the objective function as a real function to be optimized. We have to convert the uncertain objective to its crisp equivalent. In fact, there are many approaches to do so and they satisfy various requirements in different problems. These approaches are established due to some criteria including, for instance, expected value, optimistic value, pessimistic value, and etc. [19,20]. In [15], we consider making use of the optimistic value criterion. However, both the optimistic criterion and the pessimistic criterion are extreme criteria, in order to surmount the extreme cases, we may optimize the uncertain objectives applying a more flexible criterion.

Assume that \( C_t = (C_{t1}, C_{t2}, \ldots, C_{tk}) \), where \( C_{t1}, C_{t2}, \ldots, C_{tk} \) are independent canonical processes, and the symbol \( \nu^* \) denotes the transpose of a vector or a matrix. A selected coefficient \( \rho \in (0, 1) \) denoting the optimism degree, and predetermined confidence level \( \alpha \in (0, 1) \). For any \( 0 < t < T \), we present an uncertain optimal control model with Hurwicz criterion for multi-dimensional case as follows

\[
J(t, x) \equiv \sup_{u_t} \left\{ \rho F_{\sup}(x) + (1 - \rho) F_{\inf}(x) \right\}
\]

subject to

\[ dX_t = b(s, X_t, u_t)ds + \sigma(s, X_t, u_t)dc_t \quad \text{and} \quad X_T = x, \]

where, \( F = \int_t^T f(s, X_s, u_s)ds + h(T, X_T) \), and \( F_{\sup}(x) = \sup\{F|\{F \geq F\} \geq x\} \) which denotes the \( x \)-optimistic value to \( F \), \( F_{\inf}(x) = \inf\{F|\{F \leq F\} \geq x\} \) reflects the \( x \)-pessimistic value to \( F \). The vector \( X_t \) is the state vector of dimension \( n \), \( u_t \) is a control vector of dimension \( r \) subject to a constraint set \( U \). The function \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R} \) is the objective function, and \( h : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) is the function of terminal reward. In addition, \( b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n \) is a vector function, and \( \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n \times \mathbb{R}^r \) is a matrix function.

For the purpose of solving the proposed model, now we present the following principle of optimality.

**Theorem 3.1.** For any \( (t, x) \in [0, T] \times \mathbb{R}^n \), and \( \Delta t > 0 \) with \( t + \Delta t < T \), we have

\[
J(t, x) = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + f(t + \Delta t, x + \Delta X_t) + o(\Delta t) \right\},
\]

where \( x + \Delta X_t = X_{t+\Delta t} \).

**Proof.** We denote the right side of (3.2) by \( \tilde{J}(t, x) \). For arbitrary \( u_t \in U \), it follows from the definition of \( J(t, x) \) that

\[
J(t, x) \geq \rho \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + \int_t^{t+\Delta t} h(T, X_T) \right]_{\sup}(x) + (1 - \rho) \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + \int_t^{t+\Delta t} h(T, X_T) \right]_{\inf}(x),
\]

where \( u_{t\Delta t} \) and \( u_{(t+\Delta t)} \) are control vector \( u \) restricted on \([t, t + \Delta t]\) and \([t + \Delta t, T]\), respectively. Since for any \( \Delta t > 0 \),

\[
\int_t^{t+\Delta t} f(s, X_s, u_s)ds = f(t, x, u(t, x))\Delta t + o(\Delta t),
\]

we have

\[
J(t, x) \geq f(t, x, u)\Delta t + o(\Delta t) + \rho \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + h(T, X_T) \right]_{\sup}(x) + (1 - \rho) \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + h(T, X_T) \right]_{\inf}(x).
\]

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\[
J(t, x) = \sup_{u_t} \left\{ f(t, x, u_t)\Delta t + f(t + \Delta t, x + \Delta X_t) + o(\Delta t) \right\},
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where \( x + \Delta X_t = X_{t+\Delta t} \).

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\[
J(t, x) \geq \rho \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + \int_t^{t+\Delta t} h(T, X_T) \right]_{\sup}(x) + (1 - \rho) \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + \int_t^{t+\Delta t} h(T, X_T) \right]_{\inf}(x),
\]

where \( u_{t\Delta t} \) and \( u_{(t+\Delta t)} \) are control vector \( u \) restricted on \([t, t + \Delta t]\) and \([t + \Delta t, T]\), respectively. Since for any \( \Delta t > 0 \),

\[
\int_t^{t+\Delta t} f(s, X_s, u_s)ds = f(t, x, u(t, x))\Delta t + o(\Delta t),
\]

we have

\[
J(t, x) \geq f(t, x, u)\Delta t + o(\Delta t) + \rho \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + h(T, X_T) \right]_{\sup}(x) + (1 - \rho) \left[ \int_t^{t+\Delta t} f(s, X_s, u_s)ds + h(T, X_T) \right]_{\inf}(x).
\]
Taking the supremum with respect to \( u \in [0, \Delta t] \) in (3.3), we get \( f(t, x) \geq \tilde{f}(t, x) \).

On the other hand, for all \( u \) we have

\[
\rho F_{\sup}(x) + (1 - \rho)F_{\inf}(x) = f(t, x, u)\Delta t + o(\Delta t) + \rho \left[ \int_{t + \Delta t}^{T} f(s, X_s, u \mid t + \Delta t, T)ds + h(T, X_T) \right]_{\sup} \leq f(t, x, u)\Delta t + o(\Delta t) + J(t + \Delta t, x + \Delta X_t) \leq \tilde{f}(t, x).
\]

Hence, \( f(t, x) \leq \tilde{f}(t, x) \), and then \( f(t, x) = \tilde{f}(t, x) \). Theorem 3.1 is proved. \( \square \)

Theorem 3.1 tells us that, value function \( f(t, x) \) satisfies the equation comes from the principle of optimality. This equation provides a relationship among the family problem (3.1) by the value function. However, this equation is too complicated to handle. Thus, we hope to derive a simpler form of the equation for \( f(t, x) \), which is the following equation of optimality.

**Theorem 3.2.** Suppose \( f(t, x) \in C^2([0, T] \times \mathbb{R}^n) \). Then we have

\[
-J_t(t, x, u) = \sup_{u \in U}\left\{ f(t, x, u) + \nabla J(t, x, u)^T b(t, x, u) + (2\rho - 1) \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \frac{2}{a}}{a} \right) \| \nabla J(t, x, u)^T \sigma(t, x, u) \| \right\},
\]

where \( J_t(t, x, u) \) is the partial derivative of the function \( J(t, x) \) in \( t \), \( \nabla J(t, x, u) \) is the gradient of \( J(t, x) \) in \( x \), and \( \| \cdot \| \) is the 1-norm for vectors, that is, \( \| p \|_1 = \sum_{i=1}^{n} |p_i| \) for \( p = (p_1, p_2, \ldots, p_n) \).

**Proof.** Let \( H^0_s[F] = [\rho F_{\sup}(x) + (1 - \rho)F_{\inf}(x)] \). By using Taylor expansion, we get

\[
J(t + \Delta t, x + \Delta X_t) = J(t) + J_t(t, x, u)\Delta t + \nabla J(t, x)^T \Delta X_t + \frac{1}{2} J_{tt}(t, x)\Delta t^2 + \frac{1}{2} \Delta X_t^T \nabla_{xx} J(t, x) \Delta X_t + \nabla J_t(t, x) \Delta t \Delta X_t + o(\Delta t),
\]

where \( \nabla_{xx} J(t, x) \) is the Hessian matrix of \( J(t, x) \) in \( x \). Note that \( \Delta X_t = b(t, x, u)\Delta t + \sigma(t, x, u)\Delta C_t \). Substituting Eq. (3.5) into Eq. (3.2) and simplifying the resulting expression yields that

\[
0 = \sup_{u \in U}\left\{ f(t, x, u)\Delta t + J_t(t, x, u)\Delta t + \nabla J(t, x)^T b(t, x, u)\Delta t + H^0_s[\alpha \Delta C_t + \Delta C_t^T B \Delta C_t] + o(\Delta t) \right\},
\]

where

\[
a = \nabla J(t, x)^T \sigma(t, x, u) + \nabla J_t(t, x) \sigma(t, x, u)\Delta t + b(t, x, u)^T \nabla_{xx} J(t, x)^T \sigma(t, x, u)^T \Delta t,
\]

\[
B = \frac{1}{2} \sigma(t, x, u)^T \nabla_{xx} J(t, x) \sigma(t, x, u).
\]

Let \( a = (a_1, a_2, \ldots, a_k), B = (b_0)_{k+1} \). We have

\[
a \Delta C_t + \Delta C_t^T B \Delta C_t = \sum_{i=1}^{k} a_i \Delta C_i + \sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} \Delta C_i \Delta C_j.
\]

Since \( |b_{ij}| |\Delta C_i \Delta C_j| \leq \frac{1}{2} |b_{ij}| (|\Delta C_i|^2 + |\Delta C_j|^2) \), we have

\[
\sum_{i=1}^{k} \left( a_i \Delta C_i - \left( \sum_{j=1}^{k} |b_{ij}| \right) \Delta C_i^2 \right) \leq a \Delta C_t + \Delta C_t^T B \Delta C_t
\]

\[
\leq \sum_{i=1}^{k} \left( a_i \Delta C_i + \left( \sum_{j=1}^{k} |b_{ij}| \right) \Delta C_i^2 \right).
\]

Because of the independence of \( C_{i1}, C_{i2}, \ldots, C_{ik} \), we have

\[
\sum_{i=1}^{k} H^0_s \left( a_i \Delta C_i - \left( \sum_{j=1}^{k} |b_{ij}| \right) \Delta C_i^2 \right) \leq H^0_s[a \Delta C_t + \Delta C_t^T B \Delta C_t]
\]

\[
\leq \sum_{i=1}^{k} H^0_s \left( a_i \Delta C_i + \left( \sum_{j=1}^{k} |b_{ij}| \right) \Delta C_i^2 \right).
\]

By Eq. (3.6), for \( \Delta t > 0 \), any small enough \( \varepsilon > 0 \), there exists a control \( u \equiv u_{\varepsilon \Delta t} \) such that

\[
-\varepsilon \Delta t \leq \left\{ J(t, x, u)\Delta t + J_t(t, x, u)\Delta t + \nabla J(t, x)^T b(t, x, u)\Delta t + H^0_s[a \Delta C_t + \Delta C_t^T B \Delta C_t] + o(\Delta t) \right\}.
\]

Applying the second inequality from Theorem 2.2 and Inequality (A.2) in Theorem 5.1 of Appendix, we have
\(-\varepsilon \Delta t \leq f(t, x, u) \Delta t + f_i(t, x) \Delta t + \nabla_j f(t, x)^j b(t, x, u) \Delta t + (2\rho - 1) \left\{ \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \varepsilon}{\alpha - \varepsilon} \cdot \Delta t \cdot \sum_{i=1}^{k} a_i \right\} + \left( \frac{\sqrt{3}}{\pi} \ln \frac{2 - \varepsilon}{\varepsilon} \right)^2 \cdot \Delta t^2 \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} |b_{ij}| + o(\Delta t) \right\). \]

Dividing both sides of the above inequality by \(\Delta t\), and taking the supremum with respect to \(u\), we get
\[-\varepsilon \leq J_i(t, x) + \sup_{u, t} \left\{ f(t, x, u) + \nabla_j f(t, x)^j b(t, x, u) + (2\rho - 1) \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha + \varepsilon}{\alpha - \varepsilon} \cdot \|\nabla_j f(t, x)^j \sigma(t, x, u)\|_1 \right\} + h_1(\varepsilon, \Delta t) + h_2(\Delta t), \]

since \(\sum_{i=1}^{k} a_i \leq \|\nabla_j f(t, x)^j \sigma(t, x, u)\|_1\) as \(\Delta t \to 0\); where \(h_1(\varepsilon, \Delta t) \to 0\) and \(h_2(\Delta t) \to 0\) as \(\Delta t \to 0\). Letting \(\Delta t \to 0\), and then \(\varepsilon \to 0\) results in
\[-\varepsilon \leq J_i(t, x) + \sup_{u, t} \left\{ f(t, x, u) + \nabla_j f(t, x)^j b(t, x, u) + (2\rho - 1) \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \cdot \|\nabla_j f(t, x)^j \sigma(t, x, u)\|_1 \right\}. \]

On the other hand, by the third inequality from Theorem 2.2 and Inequality (A.3) in Theorem 5.1 of Appendix, applying the similar process, we can obtain
\[0 \geq J_i(t, x) + \sup_{u, t} \left\{ f(t, x, u) + \nabla_j f(t, x)^j b(t, x, u) + (2\rho - 1) \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \cdot \|\nabla_j f(t, x)^j \sigma(t, x, u)\|_1 \right\}. \]

Combining (3.7) and (3.8), we obtain the Eq. (3.4). The theorem is proved. \(\square\)

**Remark 3.1.** If we consider a discounted infinite horizon optimal control problem, we assume that the objective function \(f\), drift \(b\), disturbance \(\sigma\) are independent of time. Thus we replace \(f(s, X, u), b(s, X, u), \sigma(s, X, u)\) by \(f(X, u), b(X, u), \sigma(X, u)\), respectively. The problem is stated as follows:
\[
\begin{align*}
J(x) \equiv & \sup_{u, t} \mathcal{H}_u \left[ \int_{t}^{\infty} e^{-\gamma t} f(X_s, u) ds \right] \\
& \text{subject to} \\
& dX_s = b(X_s, u) ds + \sigma(X_s, u) dC_s, \text{ and } X_t = x.
\end{align*}
\]

At time 0, the present value of the objective is given by \(e^{-\gamma t} f(x)\), using the relations from Eq. (3.4), we obtain the present-value of (3.4),
\[
\gamma J(x) = \sup_{u, t} \left\{ f(x, u) + \nabla_j f(x)^j b(x, u) + (2\rho - 1) \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \|\nabla_j f(x)^j \sigma(x, u)\|_1 \right\}.
\]

**Example 3.1.** Consider the following optimization problem comes from the Vidale–Wolfe advertising model [21] in uncertain environments:
\[
\begin{align*}
J(0, x_0) \equiv & \max_{u, t} \mathcal{H}_u \left[ \int_{0}^{\infty} e^{-\gamma t} (\delta X_t - u^2) dt \right] \\
& \text{subject to} \\
& dX_t = [ru \sqrt{1 - X_t - kX_t}] dt + \sigma(X_t) dC_t,
\end{align*}
\]

where \(X_t \in [0, 1]\) is the fraction of market potential, \(u > 0\) denotes the rate of advertising effort, \(r > 0, k > 0, \sigma\) is a small diffusion coefficient, \(\sigma > 0, \gamma\) is a discount factor. In this case, we have \(F = \int_{0}^{\infty} e^{-\gamma t} (\delta X_t - u^2) dt\). Applying the Eq. (3.4), we obtain
\[
\gamma f = \max_{u} \left\{ (\delta x - u^2) + [ru \sqrt{1 - x - kx}] x_s + (2\rho - 1) \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left| f_x \sigma_x \right| \right\} = \max f(u),
\]

where \(L(u)\) denotes the term in the braces. Setting \(dL(u)/du = 0\) we obtain the necessary condition for optimality
\[
u = \frac{r \sqrt{1 - x}}{2} f_x(t, x).
\]

Substituting the equality into Eq. (3.11), we have
\[
\gamma f = \delta x + \frac{r^2 (1 - x)}{4} f_x^2 - kxf_x + (2\rho - 1) \left( \frac{\sqrt{3}}{\pi} \ln \frac{1 - \alpha}{\alpha} \right) \left| f_x \sigma_x \right|.
\]
We conjecture that \( J(t, x) = Px + Q \) \((P > 0)\). This gives \( J_x = P \). Using the expression in Eq. (3.12), we have the following condition for optimality

\[
\left(4\gamma P + r^2P^2 - 4\delta + 4kP\right)x + 4\gamma Q - r^2P^2 - 4P(2\rho - 1)\sqrt{3\over 2}\ln\frac{1 - \alpha}{\alpha}\sigma = 0,
\]
or

\[
4\gamma P + r^2P^2 - 4\delta + 4kP = 0, \quad \text{and} \quad 4\gamma Q - r^2P^2 - 4P(2\rho - 1)\sqrt{3\over 2}\ln\frac{1 - \alpha}{\alpha}\sigma = 0.
\]
The solution is given by

\[
P = \frac{-2(\gamma + k) + 2\sqrt{(\gamma + k)^2 + 4\delta}}{r^2} \quad \text{and} \quad Q = \frac{r^2P^2 + 4P(2\rho - 1)\sqrt{\frac{3}{2}}\ln\frac{1 - \alpha}{\alpha}\sigma}{\gamma}.
\]
The optimal decision is determined by \( u^* = \frac{r\sqrt{1 - \alpha}}{2}. \)

### 4. Application to a manufacturing technology diffusion problem

There are three phases in the life cycle of any new technology: research and development, transfer and commercialization, and operation and regeneration [22]. Investigations on the technology diffusion originated in some researches of marketing diffusion, such as Bass [23], Horsky and Simon [24]. Technology diffusion refers to the transition of technology's economic value during the transfer and operation phases of a technology life cycle. Modeling of technology diffusion must address two aspects: regularity due to the mean depletion rate of the technology’s economic value, and uncertainty owing to the disturbances occurring in technological evolution and innovation. Liu [23] studied a flexible manufacturing technology diffusion problem in a stochastic environment. If we employ uncertain differential equations as a framework to model technology diffusion problems, this flexible manufacturing technology diffusion in [25] may be solved by uncertain optimal control model with the Hurwicz criterion.

Let \( X_t \) be the potential market share at time \( t \) (state variable), and \( u \) be the proportional production level (control variable). An annual production rate can be determined as \( uX_t \). The selling price has been fairly stable at \( p \) per unit time. The unit production cost has been a function of the annual production rate, and can be calculated as \( cuX_t \), where \( c \) is a cost conversion coefficient. With constant \( \beta \) as a fixed learning percentage, the learning effect can be expressed as \( \beta X_t \). Thus, the typical drift will be

\[
b(t, X_t, u) = -\frac{uX_t}{1 + \beta X_t}.
\]
As for the disturbance \( \sigma \) of the example, The exact form depends on situations, here use \( \sigma(t, X_t, u) = \sqrt{aX_t} \), where \( a > 0 \) is scaling factor. Since the unit profit is \( (p - cuX_t) \), and the production rate is \( \frac{uX_t}{1 + \beta X_t} \), the unit profit function \( f \) is expressed as

\[
f(t, X_t, u) = (p - cuX_t) - \frac{uX_t}{1 + \beta X_t}.
\]
Let \( \gamma > 0 \) be the discount rate, \( e^{-\gamma t}h_0(k - \mu X_t) \) be the salvage value at the end time, with \( \mu > 1, k > 1 \). Then a manufacturing technology diffusion problem can be defined as to choose an appropriate control \( \bar{u} \), so that the Hurwicz weighted average total profit is maximized. The model is provided by

\[
\begin{align*}
J(0, x_0) = & \max_u \left\{ \int_0^T e^{-\gamma t} \left[ (p - cuX_t) - \frac{uX_t}{1 + \beta X_t} \right] dt + e^{-\gamma T}h_0(k - \mu X_T) \right\} \\
\text{subject to} & \quad dX_t = \left[ -\frac{uX_t}{1 + \beta X_t} \right] dt + \sqrt{aX_t} \, dC_t.
\end{align*}
\]
Conjecture that \( J_x(t, x) \geq 0 \). Then applying the equation of optimality (3.4), we have

\[
-J_t = \max_u \left\{ e^{-\gamma t}(p - cuX) - \frac{uX_t}{1 + \beta X_t} - J_x \frac{uX_t}{1 + \beta X_t} + J_x \sqrt{\alpha}(2\rho - 1) \left( \frac{\sqrt{3}}{2}\ln\frac{1 - \alpha}{\alpha}\right) \right\} \leq \max_u L(u),
\]
where \( L(u) \) represents the term enclosed by the braces. The optimal \( u \) satisfies

\[
\frac{\partial L(u)}{\partial u} = -e^{-\gamma t}cX \cdot \frac{uX}{1 + \beta X} + e^{-\gamma t}(p - cuX) \cdot \frac{X}{1 + \beta X} - J_x \frac{X}{1 + \beta X} = 0,
\]
or
\[ u = \frac{1}{2\alpha x} (p - e^{x} J_x). \]

Substituting the above result into \( \max_{x} L(u) \), we obtain

\[ -J_x = \frac{(e^{-\gamma t} p - J_x)(p - e^{x} J_x)}{4c(1 + \beta x)} + J_x \sqrt{ax} (2\rho - 1) \left( \frac{\sigma \sqrt{3}}{\pi} \ln \frac{1 - x}{x} \right). \]

The preceding is a partial differential equation. We now conjecture that \( J(t,x) = e^{-\gamma t} y(x) \), and this gives

\[ J_x = -\gamma e^{-\gamma t} y(x), \quad J_x = e^{-\gamma t} y'(x). \]

Using the last expression, denoting \( (2\rho - 1) \left( \frac{\sigma \sqrt{3}}{\pi} \ln \frac{1 - x}{x} \right) \) by parameter \( q \), we find

\[ \gamma y(x) = \frac{(p - y(x))^2}{4c(1 + \beta x)} + q \sqrt{ax} y'(x). \]

Letting \( \lambda(x) = y' \), then we have

\[ y = \lambda^2 + \frac{2[2cq(1 + \beta x)\sqrt{ax} - p] \lambda + p^2}{4\gamma c(1 + \beta x)}, \quad y' = \lambda. \]

The derivative of the right side of the first expression, should be equal to the right side of the second expression. So we get

\[ \lambda + 2cq(1 + \beta x)\sqrt{ax} - p(1 + \beta x) \frac{d\lambda}{dx} = \frac{\beta}{2(1 + \beta x)} \lambda^2 + 2\gamma c(1 + \beta x) \lambda + \left[ \frac{\beta p^2}{2(1 + \beta x)} - \frac{cq(1 + \beta x) - \beta p}{2\sqrt{ax}} \right]. \]

This differential equation is a second type of Abelian equation with respect to \( \lambda(x) \) with the following form

\[ \lambda + g(x) \frac{d\lambda}{dx} = f_2(x) \lambda^2 + f_1(x) \lambda + f_0. \]

Then solving the ordinary differential equation with terminal condition

\[ J_{X_t} = \frac{\partial}{\partial X_t} J = e^{-\gamma t} h_0(-\ln \mu \cdot \mu^{x_t}) = e^{-\gamma t} y(X_t). \]

We have

\[ y' = \lambda(x) = -g(x) + L^{-1} \left[ 1 + \left( f_1 + g' - 2f_2g \right) L dx \right]^2 + 2\int (f_0 - f_1g + f_2g^2) L^2 dx, \]

where \( L = \exp(-\int f_2 \ dx), \)

\[ = p(\beta x + 1) - 2cq\sqrt{ax}(\beta x + 1) \]

\[ + (2\beta x + 2)^2 \int_0^L \left[ 1 + \frac{1}{2} \frac{2\beta c^2(a - mp)}{15} \right] \left[ c\sqrt{ax} (8c \mu x (3\beta x + 5) + 20\beta px + 15) + px(4c\mu + \beta p) - \frac{(p - 2)p}{\beta x + 1} \right] \]

\[ + \frac{c^2 \left( \sqrt{\beta x + 1} (3a\beta q x + 4\mu \sqrt{ax}(\beta x + 1)) + 3a\sqrt{\beta q x} \text{arsinh}^{-1} (\sqrt{\beta x}) \right)^2}{18a\beta x^{x_t}} \]

where \( I_0 \) satisfies the equation \( \lambda(X_t) = h_0(-\ln \mu \cdot \mu^{x_t}). \)

The optimal proportional production level is determined by

\[ u = \frac{1}{2\alpha x} (p - \lambda(x)). \]

\[ J_x = e^{-\gamma t} \lambda(x) \] denotes the rate of the current value function.

5. Conclusion

Based on the concept of uncertain differential equation, an uncertain optimal control model with Hurwicz criterion was investigated. Applying the method of dynamic programming, a fundamental result called equation of optimality was presented for solving the model. As an application of the equation of optimality, we solved a manufacturing technology diffusion problem. In the future research direction, we are planning to investigate uncertain optimal control continuously, especially in applications.

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Appendix A

Let us give an estimation for the $\alpha$-pessimistic value of $a\xi + b\xi^2$ if $\xi$ is a normal uncertain variable ($\alpha \in (0, 1)$).

**Theorem 5.1.** Let $\xi$ be a normal uncertain variable with expected value 0 and variance $\sigma^2 (\sigma > 0)$, whose uncertainty distribution is

$$
\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}.
$$

Then for any real number $a$, and any $\varepsilon > 0$ small enough,

$$
[a\xi + b\xi^2]_{\inf}(\alpha) \geq \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} |a|\sigma + \left(\frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)^2 b\sigma^2, \tag{A.1}
$$

$$
[a\xi + b\xi^2]_{\inf}(\alpha) \leq \frac{\sqrt{3}}{\pi} \ln \frac{\alpha + \varepsilon}{1-\alpha - \varepsilon} |a|\sigma + \left(\frac{\sqrt{3}}{\pi} \ln \frac{2-\varepsilon}{\varepsilon}\right)^2 b\sigma^2, \tag{A.2}
$$

if $b > 0$; and

$$
[a\xi + b\xi^2]_{\inf}(\alpha) \geq \frac{\sqrt{3}}{\pi} \ln \frac{\alpha - \varepsilon}{1-\alpha + \varepsilon} |a|\sigma + \left(\frac{\sqrt{3}}{\pi} \ln \frac{2-\varepsilon}{\varepsilon}\right)^2 b\sigma^2, \tag{A.3}
$$

$$
[a\xi + b\xi^2]_{\inf}(\alpha) \leq \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} |a|\sigma + \left(\frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)^2 b\sigma^2, \tag{A.4}
$$

if $b < 0$; and also

$$
[a\xi + b\xi^2]_{\inf}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} |a|\sigma \tag{A.5}
$$

if $b = 0$.

**Proof.** According to the second proposition from Theorem 2.1,

If $\lambda < 0$, then $(\lambda\xi)_{\sup}(\alpha) = \lambda\xi_{\inf}(\alpha)$, and $(\lambda\xi)_{\inf}(\alpha) = \lambda\xi_{\sup}(\alpha)$.

We have

$$
[a\xi + b\xi^2]_{\inf}(\alpha) = -[-a\xi - b\xi^2]_{\sup}(\alpha).
$$

And then, via applying Theorem 2.2, the conclusions are easily proved. \qed

**References**


