A New Law of Large Numbers for Uncertain Random Variables

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Abstract
Uncertainty and randomness are two basic types of non-determinacy. Chance theory was founded for modeling complex systems with not only uncertainty but also randomness. As a mixture of randomness and uncertainty, uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables. This paper presents a law of large numbers about uncertain random variables, where the uncertain random variables are independent but not necessarily identically distributed.

Keywords: uncertain random variable, chance theory, law of large numbers, convergence in distribution

1 Introduction
Probability theory was founded by Kolmogorov [4] in 1933. We should first obtain the probability distribution via statistics before applying it in practice, then the probability distribution to make sure it is close enough to the real frequency, either of which is based on a lot of observed data. However, we sometimes have no observed data. In this case, some domain experts are invited to evaluate their belief degree of the possible events. But the belief degree usually has a much larger range than the true frequency, and probability theory is not applicable [8].


In many cases, randomness and uncertainty may be exist in a complex system, where we may have a large samples on some components but have no samples on some other components. For the first class of components, we can obtain the probability distributions of their lifetimes via statistics, but for the second class, we can only get expert’s belief degree. So the system behaves both randomly and uncertainly, and cannot be dealt with simply by probability theory or uncertainty theory. In order to describe such a system, Liu [10] pioneered a chance theory based on probability theory and uncertainty.
theory in 2013. The concepts of chance measure and uncertain random variable were proposed by Liu [10]. Uncertain random variable was proposed to model the quantities under uncertain and random condition. Meanwhile, Liu [10] defined the concepts of chance distribution, expected value and variance for uncertain random variable. Those concepts were employed to describe an uncertain random variable afterwards. Following that, Guo and Wang [2] proved a formula for calculating the variance of uncertain random variables. Sheng and Yao [16] provided some formulas to calculate the variance of uncertain random variables through chance distribution and inverse chance distribution. Yao and Gao [19] verified a law of large numbers for iid uncertain random variables.

In this paper, we will prove a new law of large numbers for uncertain random variables of independent but not necessarily identically distributed. This paper structure as follows. The next section is intended to introduce some concepts and theorems of uncertain theory and chance theory. A new law of large numbers is proved about uncertain random variables in Section 3. At last, some conclusions are given in Section 4.

2 Preliminary

In this section, we will introduce some useful definitions and theorems about uncertain variable and uncertain random variable.

2.1 Uncertainty Theory

Definition 1. (Liu [5]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( \mathcal{M} : \mathcal{L} \rightarrow [0, 1] \) is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) \( \mathcal{M}\{\Gamma\} = 1 \) for the universal set \( \Gamma \).

Axiom 2: (Duality Axiom) \( \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1 \) for any event \( \Lambda \).

Axiom 3: (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \cdots \), we have

\[
\mathcal{M}\{ \bigcup_{i=1}^{\infty} \Lambda_i \} \leq \sum_{i=1}^{\infty} \mathcal{M}\{ \Lambda_i \}.
\]

Besides, the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) is defined by Liu [6] as follows,

Axiom 4: (Product Axiom) Let \( (\Gamma_k, \mathcal{L}_k, \mathcal{M}_k) \) be uncertainty spaces for \( k = 1, 2, \cdots \). Then the product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying

\[
\mathcal{M}\{ \prod_{i=1}^{\infty} \Lambda_k \} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{ \Lambda_k \}
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \cdots \), respectively.

An uncertain variable is essentially a measurable function from an uncertainty space to the set of real numbers. In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2. (Liu [5]) Let \( \xi \) be an uncertain variable. Then its uncertainty distribution is defined by

\[
\Phi(x) = \mathcal{M}\{\xi \leq x\}
\]

for any real number \( x \).

An uncertainty distribution \( \Phi \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \). Inverse uncertainty distribution plays an important role in the operation of independent uncertain variables.
Definition 3. (Liu [6]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if 

$$
M \left\{ \bigcap_{i=1}^{m} (\xi_i \in B_i) \right\} = \bigwedge_{k=1}^{m} M \{ \xi_k \in B_k \}
$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

The inverse distribution of a strictly monotone function of uncertain variables can be obtained by the following theorem.

Theorem 1. (Liu [6]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution 

$$
\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).
$$

The distribution of a strictly increasing function of uncertain variables can be obtained by the following theorem.

Theorem 2. (Liu [7]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f$ is strictly increasing function, then 

$$
\xi = f(\xi_1, \xi_2, \cdots, \xi_n)
$$

is an uncertain variable with uncertainty distribution 

$$
\Phi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \min_{1 \leq i \leq n} \Phi_i(x_i).
$$

Definition 4. (Liu [5]) The expected value of an uncertain variable $\xi$ is defined by 

$$
E[\xi] = \int_{-\infty}^{+\infty} \Phi(\xi) d\xi - \int_{-\infty}^{0} \Phi(\xi) d\xi
$$

provided that at least one of the two integrals is finite.

Theorem 3. (Liu [5]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. If the expected value exists, then 

$$
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^{0} \Phi(x) dx.
$$

In 2010, Liu [6] first introduced a formula expected value by inverse uncertainty distribution, that is 

$$
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha.
$$

Liu and Ha [12] proposed a generalized formula for expected value by inverse uncertainty distribution.

Theorem 4. (Liu and Ha [12]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value 

$$
E[\xi] = \int_{0}^{1} f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) d\alpha.
$$


2.2 Chance Theory

The product \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) is called a chance space, in which \((\Gamma, \mathcal{L}, \mathcal{M})\) is an uncertainty space and \((\Omega, \mathcal{A}, \Pr)\) is a probability space.

**Definition 5.** (Liu [10]) Let \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space, and let \(\Theta \in \mathcal{L} \times \mathcal{A}\) be an uncertain random event. Then the chance measure of \(\Theta\) is defined as
\[
\text{Ch}\{\Theta\} = \int_{0}^{1} \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\} \, dr.
\]

Liu [10] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is (i) \(\text{Ch}\{\Gamma \times \Omega\} = 1\); (ii) \(\text{Ch}\{\Theta\} + \text{Ch}\{\Theta^c\} = 1\) for any event \(\Theta\); (iii) \(\text{Ch}\{\Theta_1\} \leq \text{Ch}\{\Theta_2\}\) for any real number set \(\Theta_1 \subset \Theta_2\).

Besides, Hou [3] proved the subadditivity of chance measure, that is,
\[
\text{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} \text{Ch}\{\Theta_i\}
\]
for a sequence of events \(\Theta_1, \Theta_2, \cdots\).

**Definition 6.** (Liu [10]) An uncertain random variable is a measurable function \(\xi\) from a chance space \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) to the set of real numbers, i.e., \(\{\xi \in B\}\) is an event for any Borel set \(B\).

Random variables and uncertain variables can be regarded as special cases of uncertain random variables. Let \(\eta\) be a random variable, \(\tau\) be an uncertain variable. Then both \(\eta + \tau\) and \(\eta \times \tau\) are uncertain random variables.

**Theorem 5.** (Liu [10]) Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a measurable function, and \(\xi_1, \xi_2, \cdots, \xi_n\) uncertain random variables on the chance space \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\). Then \(\xi = f(\xi_1, \xi_2, \cdots, \xi_n)\) is an uncertain random variable determined by
\[
\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega))
\]
for all \((\gamma, \omega) \in \Gamma \times \Omega\).

To calculate the chance measure, Liu [10] presented a definition of chance distribution.

**Definition 7.** (Liu [10]) Let \(\xi\) be an uncertain random variable. Then its chance distribution is defined by
\[
\Phi(x) = \text{Ch}\{\xi \leq x\}
\]
for any \(x \in \mathbb{R}\).

The chance distribution of a random variable is just its probability distribution, and the chance distribution of an uncertain variable is just its uncertainty distribution.

**Theorem 6.** (Liu [10]) Let \(\eta_1, \eta_2, \cdots, \eta_m\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \cdots, \Psi_m\), respectively, and let \(\tau_1, \tau_2, \cdots, \tau_n\) be uncertain variables. Then the uncertain random variable
\[
\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)
\]
has a chance distribution
\[
\Phi(x) = \int_{\mathbb{R}^m} F(x, y_1, \cdots, y_m) \, d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
where \(F(x, y_1, \cdots, y_m)\) is the uncertainty distribution of uncertain variable \(f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)\) for any real numbers \(y_1, y_2, \cdots, y_m\).
Definition 8. (Liu [10]) Let $\xi$ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq r\}dr - \int_{-\infty}^{0} \text{Ch}\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.

Let $\Phi$ denote the chance distribution of $\xi$. Liu [11] proved a formula to calculate the expected value of uncertain random variable with chance distribution, that is,

$$E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx.$$

For a random variable $\eta$ and an uncertain variable $\tau$, Liu [11] proved that $E[\eta + \tau] = E[\eta] + E[\tau]$ and $E[\eta \times \tau] = E[\eta] \times E[\tau]$. In fact, we have the following theorem about the expected value of uncertain random variables.

Theorem 7. (Liu [11]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables (not necessarily independent). Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has an expected value

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

where

$$E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]$$

is the expected value of the uncertain variable

$$f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$$

for any real numbers $y_1, y_2, \cdots, y_m$.

Definition 9. (Yao and Gao [19]) Suppose that $\Phi, \Phi_1, \Phi_2, \cdots$ are the chance distributions of uncertain random variables $\xi, \xi_1, \xi_2, \cdots$, respectively. Then $\{\xi_i\}$ is said to converge in distribution to $\xi$ if

$$\lim_{i \to \infty} \Phi_i(x) = \Phi(x)$$

for every number $x \in \mathbb{R}$ at which $\Phi$ is continuous.

Theorem 8. (Law of Large Numbers, Yao and Gao [19]) Let $\eta_1, \eta_2, \cdots$ be a sequence of iid random variables with a probability distribution $\Phi$, and $\tau_1, \tau_2, \cdots$ be a sequence of iid uncertain variables. Define

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)$$

where $f$ is a strictly monotone function. Then $\{S_n/n\}$ converges in distribution to

$$\int_{-\infty}^{+\infty} f(x, \tau_1)d\Phi(x) \quad \text{as} \quad n \to \infty.$$
3 Law of Large Numbers for Uncertain Random Variables

Yao and Gao [19] verified the law of large numbers for iid uncertain random variables. In this section we will proved a law of large numbers for uncertain random variables, where the uncertain random variables are independent but not necessarily identically distributed.

For the proof of law of large numbers for uncertain random variables, where a sequence uncertain random variables is independent but not necessarily identically distributed, we need the following lemma.

Lemma 1. (Kolmogorov’s Stronger Law of Large Numbers [14]) Let \( \eta_1, \eta_2, \cdots \) be independent but not necessarily identically distributed random variables, and let there be positive numbers \( b_n \) such that \( b_n \uparrow \infty \) and
\[
\sum_{n=1}^{\infty} \frac{\text{Var}[\eta_n]}{b_n^2} < \infty.
\]
Then
\[
\frac{1}{b_n} \sum_{i=1}^{n} \eta_i - \frac{1}{b_n} \sum_{i=1}^{n} E \eta_i \rightarrow 0 \quad (a.s.).
\]
In particular, if
\[
\sum_{n=1}^{\infty} \frac{\text{Var}[\eta_n]}{n^2} < \infty
\]
then
\[
\frac{1}{n} \sum_{i=1}^{n} \eta_i - \frac{1}{n} \sum_{i=1}^{n} E \eta_i \rightarrow 0 \quad (a.s.).
\]

Theorem 9. (Law of Large Numbers for Uncertain Random Variables) Let \( \eta_1, \eta_2, \cdots \) be a sequence of independent random variables with probability distributions \( \Phi_1(x), \Phi_2(x), \cdots, \) respectively, let \( \tau_1, \tau_2, \cdots \) be a sequence of independent uncertain variables with uncertainty distributions \( \Psi_1(y), \Psi_2(y), \cdots, \) respectively. Define \( S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n) \) where \( f \) is a strictly monotone function with respect to \( \tau_i, i = 1, 2, \cdots. \) If
\[
\sum_{n=1}^{\infty} \frac{\text{Var}[f(\eta_n, y)]}{n^2} < \infty
\]
for any \( y \in \mathbb{R} \) and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x)
\]
exists. Then \( \{S_n/n\} \) converges in distribution to
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \quad \text{as} \quad n \to \infty.
\]

Proof: According to the definition of convergence in distribution, it is equivalent to prove
\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \right\}
\]=\text{Ch} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \right\}
=\mathcal{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \right\}
for any real number \( y \) with

\[
\lim_{u \to y} \mathcal{M} \left\{ \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i) \right\} \leq \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(x, u) \right\} = \mathcal{M} \left\{ \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i) \right\} \leq \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \right\}.
\]

The proof breaks into two cases according to the monotonicity of the function \( f \).

Case 1: Assume \( f \) is a strictly increasing function with respect to \( \tau_i \). Since \( \tau_1, \tau_2, \cdots \) are a sequence of independent uncertain variables with uncertainty distributions \( \Psi_1(y), \Psi_2(y), \cdots \), respectively, we obtain that \( f(x, \tau_1), f(x, \tau_2), \cdots \) are a sequence of independent uncertain variables for any \( x \in \mathbb{R} \) and there exists a function \( g \) such that uncertainty distributions of \( f(x, \tau_1), f(x, \tau_2), \cdots \) are

\[
\mathcal{M}\{f(\eta_i = x, \tau_i) \leq y\} = \Psi_i(g(x, y)), \quad \forall x \in \mathbb{R}, i = 1, 2, \cdots
\]

we obtain also that

\[
\mathcal{M}\{f(\eta_i = x, \tau_i) \leq f(\eta_i = x, y)\} = \Psi_i(g(f(x, y))) = \Psi_i(y), \quad \forall x \in \mathbb{R}, i = 1, 2, \cdots
\]

then we have

\[
\mathcal{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, \tau_i) \right\} \leq \frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, y) \right\} = \sup_{1 \leq i \leq n} \min \Psi_i(g(x, y))
\]

by Theorem 2. Then we have

\[
\mathcal{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, \tau_i) \leq \frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, y) \right\} = \sup_{1 \leq i \leq n} \min \Psi_i(g(f(x, y)))
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, y) = f(x, y)
\]

\[
\min \Psi_i(y_i) = \sup_{1 \leq i \leq n} \Psi_i(y_i).
\]

For simplicity, we write

\[
\frac{1}{n} \sum_{i=1}^{n} f(\eta_i = x, y) = f(x, y)
\]

Since \( \eta_1, \eta_2, \cdots \) are a sequence of independent random variables with probability distributions \( \Phi_1(x), \Phi_2(x), \cdots \), respectively, we obtain that \( f(\eta_1, y), f(\eta_2, y), \cdots \) are a sequence of independent random variables for any \( y \in \mathbb{R} \), then we have

\[
\frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) - \frac{1}{n} \sum_{i=1}^{n} f(x, y) \right\} d\Phi_i(x) \to 0 \quad (a.s.)
\]
by Lemma 1. Then we obtain also that,

\[
\lim_{n \to \infty} \Psi(n, y) = \lim_{n \to \infty} \mathbb{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \right\}
\]

\[
= \mathbb{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \right\}
\]

\[
= \mathbb{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, \tau_i) d\Phi_i(x) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) \right\} \quad (a.s.)
\]

and for any \( \varepsilon > 0 \), there exists a positive number \( N_1 \) such that

\[
\Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right\} \geq 1 - \varepsilon,
\]

thus

\[
\text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right\}
\]

\[
= \int_{0}^{1} \Pr \left\{ \mathbb{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right\} \geq r \right\} dr
\]

\[
\geq \int_{0}^{1} \Pr \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right) \right.
\]

\[
\cap \left( \mathbb{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right\} \geq r \right) \left\} \right. 
\]

\[
\geq \int_{0}^{1} \Pr \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right) \right.
\]

\[
\cap \left( \mathbb{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) \leq \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \right\} \geq r \right) \left\} \right. 
\]

\[
= \int_{0}^{\Psi(n, y)} \Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) + \varepsilon \right\} dr
\]

\[
\geq \Psi(n, y)(1 - \varepsilon)
\]

for any \( n \geq N_1 \). Meanwhile, there exists a positive number \( N_2 \) such that

\[
\Pr \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f(x, y) d\Phi_i(x) - \varepsilon \right\} \geq 1 - \varepsilon,
\]
thus
\[
\begin{align*}
\text{Ch} \left\{ \frac{S_n}{n} > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right\} \\
= \int_{0}^{1} \text{Pr} \left\{ M \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right\} > r \right\} \text{d}r \\
\geq \int_{0}^{1} \text{Pr} \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right) \\
\quad \cap \left( M \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, \tau_i) > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right\} \geq r \right) \right\} \text{d}r \\
\geq \int_{0}^{1-\Psi(n, y)} \text{Pr} \left\{ \frac{1}{n} \sum_{i=1}^{n} f(\eta_i, y) > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right\} \text{d}r \\
\geq (1 - \Psi(n, y))(1 - \varepsilon)
\end{align*}
\]
\]
for any \( n \geq N_2 \). By the duality of chance measure, we have
\[
\text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) - \varepsilon \right\} \\
\leq 1 - (1 - \Psi(n, y))(1 - \varepsilon).
\]

Since \( \Psi(n, y)(1 - \varepsilon) \to \Psi(n, y) \) and \( 1 - (1 - \Psi(n, y))(1 - \varepsilon) \to \Psi(n, y) \) as \( \varepsilon \to 0 \), there exists a positive number \( N = \max\{N_1, N_2\} \), for any \( n \geq N \), we have
\[
\Psi(n, y) \leq \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) \right\} \leq \Psi(n, y).
\]

As a result, we obtain
\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) \right\} \\
= \lim_{n \to \infty} \Psi(n, y).
\]

So we have
\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) \right\} \\
= \lim_{n \to \infty} \Psi(n, y) \\
= \mathcal{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i) \text{d}\Phi_i(x) \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, y) \text{d}\Phi_i(x) \right\}.
\]

Case 1 is thus proved.

Case 2: Assume \( f \) is a strictly decreasing function. Then \( -f \) is a strictly increasing function. By Case 1,
which leads to

$$\lim_{n \to \infty} \text{Ch} \left\{ -\frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} f(x, y) d\Phi_i(x) \right\}$$

$$= \mathcal{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} f(x, y) d\Phi_i(x) \right\} ,$$

which apparently. This completes the proof of the theorem.

**Remark 3.** For a sequence of independent random variables $\eta_1, \eta_2, \cdots$, write $S_n = \eta_1 + \eta_2 + \cdots + \eta_n$ for $n \geq 1$. Then it follows from Theorem 9 that $\{S_n/n\}$ converges in distribution to $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \eta_i$.

**Remark 4.** For a sequence of independent uncertain variables $\tau_1, \tau_2, \cdots$, write $S_n = \tau_1 + \tau_2 + \cdots + \tau_n$ for $n \geq 1$. Then it follows from Theorem 9 that $\{S_n/n\}$ converges in distribution to $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \tau_i$.

If a sequence random variables $\eta_1, \eta_2, \cdots$ are iid. Then Remark 3 is just Remark 1. If a sequence uncertain variables $\tau_1, \tau_2, \cdots$ are iid. Then Remark 4 is just Remark 2.

**Example 1.** Let $\eta_1, \eta_2, \cdots$ be a sequence of independent positive random variables with probability distribution $\Phi_1(x), \Phi_2(x), \cdots$, respectively, let $\tau_1, \tau_2, \cdots$ be a sequence of independent uncertain variables with uncertainty distribution $\Psi_1(y), \Psi_2(y), \cdots$, respectively. Define

$$S_n = (\eta_1 + \tau_1) + (\eta_2 + \tau_2) + \cdots + (\eta_n + \tau_n).$$

If

$$\sum_{n=1}^{\infty} \frac{\text{Var}[\eta_n]}{n^2} < \infty$$

for any $y \in \mathbb{R}$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} (E[\eta_i] + \tau_i)$$

exists. Then $\{S_n/n\}$ converges in distribution to

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} f(x, y) d\Phi_i(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} (E[\eta_i] + \tau_i).$$

**Example 2.** Let $\eta_1, \eta_2, \cdots$ be a sequence of independent positive random variables with probability distribution $\Phi_1(x), \Phi_2(x), \cdots$, respectively, let $\tau_1, \tau_2, \cdots$ be a sequence of independent uncertain variables with uncertainty distribution $\Psi_1(y), \Psi_2(y), \cdots$, respectively. Define

$$S_n = (\eta_1 \tau_1) + (\eta_2 \tau_2) + \cdots + (\eta_n \tau_n).$$
If
\[ \sum_{n=1}^{\infty} \frac{Var[\eta_n]}{n^2} < \infty \]
for any \( y \in \mathbb{R} \) and
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (E[\eta_i] \tau_i) \]
exists. Then \( \{S_n/n\} \) converges in distribution to
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} (x \tau_i) d\Phi_i(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (E[\eta_i] \tau_i). \]

4 Conclusions

This paper proved a law of large numbers about uncertain random variables, where a sequence uncertain random variables is independent but not necessarily identically distributed. The conclusions is average of uncertain random variables converges in distribution to limit of average of expected value of random variables and uncertain variables.

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References


