Path Optimality Conditions for Minimum Spanning Tree Problem with Uncertain Edge Weights

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This paper investigates the uncertain minimum spanning tree (UMST) problem where the edge weights are assumed to be uncertain variables. In order to propose effective solving methods for the UMST problem, path optimality conditions as well as some equivalent definitions for two commonly used types of UMST, namely, uncertain expected minimum spanning tree (expected UMST) and uncertain α-minimum spanning tree (α-UMST), are discussed. It is shown that both the expected UMST problem and the α-UMST problem can be transformed into an equivalent classical minimum spanning tree problem on a corresponding deterministic graph, which leads to effective algorithms with low computational complexity. Furthermore, the notion of uncertain most minimum spanning tree (most UMST) is initiated for an uncertain graph, and then the equivalent relationship between the α-UMST and the most UMST is proved. Numerical examples are presented as well for illustration.

Keywords: Minimum spanning tree; uncertainty theory; path optimality condition.

1. Introduction

The minimum spanning tree (MST) problem is to find a tree that connects all the vertices in a graph with the minimum total weight. The weight assigned to each edge of the graph can represent cost, time, length and so on depending on the context. The application of MST extensively exists in the areas of statistical cluster analysis,1 image processing,2 communication network,3 etc. For instance, in network routing protocols, the minimum cost spanning tree is one of the most effective methods to broadcast the massages from a source node to a set of destinations.

In the classical MST problem, the edge weights associated to a graph are usually assumed to be crisp numbers. For this deterministic case, the MST problem can

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be solved in polynomial time by some well-known algorithms, such as the Kruskal algorithm\(^4\) and the Prim algorithm.\(^5\) However, the edge weights are not always deterministic in real applications. For example, the links in a communication network may be affected by collisions, congestions, interferences or some other factors. Consequently, the bandwidths of these links, which are denoted by edge weights in the graph of the communication network, are nondeterministic. In this case, some researchers believed that these nondeterministic phenomena conform to randomness or fuzziness, and hence the probability theory or the fuzzy set theory was introduced into the MST problem.

Ishii\( et \ al.\)^6 first proposed the stochastic spanning tree problem, where the weights are expressed as random variables. Following that, Ishii and Matsutomi\(^7\) presented a polynomial time algorithm to solve the stochastic spanning tree problem when the parameters of probability distributions of the edge weights are unknown. This algorithm estimates the unknown parameters by applying a confidence region from stochastic data. Torkestani and Meybodi\(^8\) proposed a learning automata-based heuristic algorithm which significantly decreases the rate of unnecessary samples to solve the stochastic MST problem with unknown probability distributions of weights. Furthermore, Dhandhere\( et \ al.\)^9 and Swamy and Shmoys\(^10\) discussed the two-stage stochastic MST problems.

As for the minimum spanning tree problem under fuzziness, Itoh and Ishii\(^11\) first formulated an MST problem with fuzzy edge weights as a chance-constrained programming based on the necessity measure. Following that, three approaches based on the overall existence ranking index for ranking fuzzy edge weights of spanning trees were presented by Chang and Lee.\(^12\) Almeida\( et \ al.\)^13 studied the MST problem with fuzzy parameters and proposed an exact algorithm as well as a special genetic algorithm based on the fuzzy set theory and the probability theory. Janiak and Kasperski\(^14\) applied the possibility theory to characterize the optimality of edges of the graph where the edge costs are specified as fuzzy intervals. Subsequently, based on the credibility theory founded by Liu,\(^15\) Gao and Lu\(^16\) considered the fuzzy quadratic MST problem, and formulated it as the expected value model, the chance-constrained programming and the dependent-chance programming according to different decision criteria.

However, it has been shown that it is inappropriate to describe the nondeterministic phenomena as randomness or fuzziness in many scenarios, particularly those involving the linguistic ambiguity and subjective estimation, since both the probability theory and the fuzzy set theory may lead to counterintuitive results (see Ref.\(^17\) for details). In the MST problem, when no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree about the unknown state of nature. The belief degrees evaluated by some domain experts may have much bigger variance than the real frequency. In this case, the probability theory or the fuzzy set theory is no longer suitable, whereas uncertainty theory proposed by Liu\(^17\) provides an alternative appropriate framework to deal with it. Based on uncertainty theory, Peng and Li\(^18\) considered...
the uncertain minimum spanning tree (UMST) problem where the edge weights are assumed to be uncertain variables, and proposed three types of UMST, namely, the uncertain expected minimum spanning tree (expected UMST), the uncertain \( \alpha \)-minimum spanning tree (\( \alpha \)-UMST), and the uncertain distribution minimum spanning tree. After that, Peng and Zhang\(^{19}\) reviewed the recent advances in uncertain network optimization, and presented some general uncertain network optimization models based on uncertain programming. Recently, Zhang et al.\(^{20}\) studied the inverse minimum spanning tree problem, and Zhou et al.\(^{21}\) introduced the uncertain quadratic minimum spanning tree problem.

Although Peng and Li\(^{18}\) proposed the concepts of three types of UMST as well as a 99-table algorithm for finding the inverse uncertainty distribution of uncertain spanning tree, effective solving methods for the UMST problem are not discussed. In this paper, we make a further study of the MST problem with uncertain edge weights. In order to propose effective solving methods, path optimality conditions as well as some equivalent definitions for UMST are discussed. Furthermore, a new type of UMST, the so-called uncertain most minimum spanning tree (most UMST), is also initiated. Solving methods for the most UMST problem is presented as well.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts in the uncertainty theory. Section 3 describes the classical MST problem and the definitions of expected UMST and \( \alpha \)-UMST. In Secs. 4 and 5, we discuss the path optimality conditions and equivalent definitions of expected UMST and \( \alpha \)-UMST, respectively. Section 6 proposes the concept of most UMST, and then discusses the relationship between it and \( \alpha \)-UMST. Finally, numerical examples are given in Sec. 7 for illustration.

2. Preliminaries

Uncertainty theory, founded by Liu,\(^{17,22}\) is an efficient tool to deal with nondeterministic information, especially expert data and subjective estimations. By now, it has been applied to many areas, and brought many branches such as uncertain programming, uncertain statistics, uncertain logic, uncertain inference, uncertain process, and uncertain finance.\(^{23-30}\)

In this section, we introduce some fundamental concepts and properties of the uncertainty theory, which will be used throughout this paper.

**Definition 1.** (Liu\(^{17}\)) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( M: \mathcal{L} \to [0,1] \) is called an uncertain measure if it satisfies the following axioms:

**Axiom 1.** (Normality Axiom) \( M(\Gamma) = 1 \) for the universal set \( \Gamma \);

**Axiom 2.** (Duality Axiom) \( M(\Lambda) + M(\Lambda^c) = 1 \) for any event \( \Lambda \);

**Axiom 3.** (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
M \left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i)
\]
Besides, the triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space. Moreover, let 
\((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \(k = 1, 2, \ldots\). Denote
\[\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots, \quad \mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots.\] (2)

Then the product uncertain measure \(\mathcal{M}\) on the product \(\sigma\)-algebra \(\mathcal{L}\) is defined by

**Axiom 4.** (Product Axiom) Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \(k = 1, 2, \ldots\). The product uncertain measure \(\mathcal{M}\) is an uncertain measure satisfying

\[\mathcal{M}\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}\] (3)

where \(\Lambda_k\) are arbitrarily chosen events from \(\mathcal{L}_k\) for \(k = 1, 2, \ldots\), respectively.

**Definition 2.** (Liu\textsuperscript{17}) An uncertain variable is a measurable function \(\xi\) from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers, i.e., for any Borel set \(B\) of real numbers, the set

\[\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}\] (4)

is an event.

**Definition 3.** (Liu\textsuperscript{17}) Let \(\xi\) be an uncertain variable. Its uncertainty distribution is defined by

\[\Phi(x) = \mathcal{M}\{\xi \leq x\}\] (5)

for any real number \(x\).

For example, an uncertain variable \(\xi\) is called *linear* if it has a linear uncertainty distribution (see Fig. 1)

\[\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a, \\
(x - a)/(b - a), & \text{if } a \leq x \leq b, \\
1, & \text{if } x \geq b,
\end{cases}\] (6)

denoted by \(\mathcal{L}(a, b)\), where \(a\) and \(b\) are real numbers with \(a < b\).

An uncertain variable \(\xi\) is called *zigzag* if it has a zigzag uncertainty distribution (see Fig. 2)

\[\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a, \\
(x - a)/2(b - a), & \text{if } a \leq x \leq \frac{b + c}{2}, \\
(x + c - 2b)/2(c - b), & \text{if } \frac{b + c}{2} \leq x \leq c, \\
1, & \text{if } x \geq c.
\end{cases}\] (7)

denoted by \(\mathcal{Z}(a, b, c)\), where \(a, b\) and \(c\) are real numbers with \(a < b < c\).
An uncertainty distribution $\Phi$ is said to be \textit{regular} if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. Note that the inverse credibility distribution $\Phi^{-1}$ is well-defined on the open interval $(0, 1)$. If required, we may extend the domain to $[0, 1]$ via

$$\Phi^{-1}(0) = \lim_{\alpha \to 0^+} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \to 1^-} \Phi^{-1}(\alpha).$$

(8)

It is clear that the linear and zigzag uncertainty distributions are both regular. The inverse uncertainty distribution of a linear uncertain variable $\xi \sim L(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b,$$

(9)

while the inverse uncertainty distribution of a zigzag uncertain variable $\xi \sim Z(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} a + 2(b - a)\alpha, & \text{if } \alpha \leq 0.5, \\ 2b - c + 2(c - b)\alpha, & \text{if } \alpha \geq 0.5. \end{cases}$$

(10)

**Definition 4.** (Liu\textsuperscript{26,31}) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathbb{M}\left\{\bigcap_{i=1}^n \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^n \mathbb{M}\{\xi_i \in B_i\}$$

(11)

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

**Theorem 1.** (Liu\textsuperscript{22}) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous and strictly increasing function. Then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

(12)

**Definition 5.** (Liu\textsuperscript{17}) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_0^{+\infty} \mathbb{M}\{\xi \geq r\} \, dr - \int_{-\infty}^0 \mathbb{M}\{\xi \leq r\} \, dr$$

(13)

provided that at least one of the two integrals is finite.
Theorem 2. (Liu\cite{22}) Let $\xi$ and $\eta$ be independent uncertain variables with finite expected values. Then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

(14)

3. Problem Description

In this section, we briefly review the classical minimum spanning tree problem and then describe the uncertain minimum spanning tree problem with uncertain edge weights.

3.1. Classical minimum spanning tree problem

Let $G = (V,E,w)$ denote a connected graph consisting of the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, the edge set $E = \{e_1, e_2, \ldots, e_m\}$, and the edge weight vector $w = (w_1, w_2, \ldots, w_m)^T$. A spanning tree $T = (V, S)$ of $G$ is a connected acyclic subgraph containing all vertices, where $S$ is the set of edges contained in $T$. For simplicity, we denote a spanning tree $T$ by its edge set $S$ in our paper. Then the classical MST problem is to find a spanning tree with the total edge weight less than or equal to the weight of every other spanning tree.

Definition 6. (Minimum Spanning Tree) Given a connected graph $G = (V,E,w)$, a spanning tree $T^0$ is said to be a minimum spanning tree if

$$\sum_{e_i \in T^0} w_i \leq \sum_{e_j \in T} w_j$$

(15)

holds for any spanning tree $T$.

In order to discuss the properties of a minimum spanning tree, more concepts are given as follows. We refer to the edges in a spanning tree $T$ as tree edges, and the edges not in $T$ are non-tree edges. It is well known that a spanning tree induces a unique path between every pair of vertices. Especially, for any non-tree edge $e_j$, there must exist a unique path consisting of tree edges between the vertices of $e_j$, which is called the tree path of edge $e_j$ and denoted by $P_j$. Following from the concepts of tree edges, non-tree edges and tree path, an equivalent condition of minimum spanning tree, known as the path optimality condition, was presented by Ahuja et al.\cite{32} as follows.

Theorem 3. (Ahuja et al.\cite{32}) Given a connected graph $G = (V,E,w)$, a spanning tree $T^0$ is a minimum spanning tree if and only if

$$w_i - w_j \leq 0, \ e_j \in E \setminus T^0, e_i \in P_j$$

(16)

where $E \setminus T^0$ is the set of non-tree edges, and $P_j$ is the corresponding tree path of edge $e_j$. 
Example 1. A graph with 6 vertices and 10 edges is shown in Fig. 3, where \( w_i \) denotes the edge weight of \( e_i \). The solid line represents a spanning tree and denote it by \( T^0 \). Then the set of non-tree edges of \( T^0 \) is \( E \setminus T^0 = \{e_3, e_6, e_8, e_9, e_{10}\} \), and the tree path of non-tree edge \( e_9 \) is \( P_9 = \{e_1, e_4, e_7\} \). Suppose that \( T^0 \) is a minimum spanning tree. Then from Theorem 3 we know that \( w_1, w_4 \) and \( w_7 \) are all less than or equal to \( w_9 \). Otherwise, we can produce a new spanning tree \( T' \) with less edge weight by replacing the bigger one in \( \{e_1, e_4, e_7\} \) with \( e_9 \).

![Fig. 3. An example of the classical minimum spanning tree problem.](image)

3.2. Uncertain minimum spanning tree problem

Due to some economic reasons or technical difficulties, we often lack observed data, and the edge weights of a graph may not be precisely known. In this case, as mentioned in the section of Introduction, we have to invite some domain experts to evaluate the belief degree about the unknown state, which makes the probability theory or the fuzzy set theory is no longer inappropriate to model the problem, whereas the uncertainty theory provides an alternative appropriate framework to deal with it. Therefore, we assume the nondeterministic edge weights to be uncertain variables \( \xi_i, i = 1, 2, \ldots, m \), and consider the uncertain version of the classical MST problem on an uncertain graph, called the uncertain minimum spanning tree problem in this paper.

The UMST problem also extensively exists in real applications. Taking the communication network for instance, if the communication costs among centers are nondeterministic due to collisions, congestions or some other reasons, then finding the minimum cost spanning tree becomes a UMST problem.

For convenience, let \( \tilde{G} = (V, E, \xi) \) denote the uncertain graph consisting of the vertex set \( V = \{v_1, v_2, \ldots, v_n\} \), the edge set \( E = \{e_1, e_2, \ldots, e_m\} \), and the edge weight vector \( \xi = (\xi_1, \xi_2, \ldots, \xi_m)^T \). In general, we assume that throughout this paper all the uncertain variables \( \xi_i, i = 1, 2, \ldots, m \), are independent and with regular uncertainty distributions, which is appropriate in real applications.

In the UMST problem, we denote the weight of a spanning tree \( T \) as \( W(T, \xi) \), i.e.,

\[
W(T, \xi) = \sum_{e_i \in T} \xi_i. \tag{17}
\]
Since $\xi_i$, $i = 1, 2, \ldots, m$, are uncertain variables, $W(T, \xi)$ is an uncertain variable as well. Consequently, Definition 6 becomes useless for the uncertain version of the minimum spanning tree, due to the uncertainty of the weight $W(T, \xi)$ of a spanning tree. In this case, Peng and Li\textsuperscript{18} gave two definitions called uncertain expected minimum spanning tree and uncertain $\alpha$-minimum spanning tree as follows.

**Definition 7.** (Peng and Li,\textsuperscript{18} Expected UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is called an uncertain expected minimum spanning tree if

$$E[W(T^0, \xi)] \leq E[W(T, \xi)]$$  \hspace{1cm} (18)

holds for any spanning tree $T$, where $E[W(T^0, \xi)]$ and $E[W(T, \xi)]$ are the expected values of weights of spanning trees $T^0$ and $T$, respectively, and $E[W(T^0, \xi)]$ is called the expected minimum spanning tree weight.

**Definition 8.** (Peng and Li,\textsuperscript{18} $\alpha$-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ and a predetermined confidence level $\alpha \in (0, 1]$, a spanning tree $T^0$ is called an uncertain $\alpha$-minimum spanning tree if

$$\min\{\omega \mid M\{W(T^0, \xi) \leq \omega \} \geq \alpha\} \leq \min\{\omega \mid M\{W(T, \xi) \leq \omega \} \geq \alpha\}$$  \hspace{1cm} (19)

holds for any spanning tree $T$, and $\min\{\omega \mid M\{W(T^0, \xi) \leq \omega \} \geq \alpha\}$ is called the $\alpha$-minimum spanning tree weight, denoted by $W_\alpha(T^0, \xi)$.

Definition 7 means that the decision-maker wants to minimize the expected value of the uncertain weight. Definition 8 shows that the decision-maker sets a confidence level $\alpha$ as an appropriate safety margin, and hopes to minimize a critical value $\omega$ such that $M\{W(T^0, \xi) \leq \omega\} \geq \alpha$. As Peng and Li\textsuperscript{18} only presented the concepts of expected UMST and $\alpha$-UMST, and the solving methods to find out the uncertain minimum spanning trees are not proposed, we further discuss the solving methods by proving the corresponding path optimality conditions for expected UMST and $\alpha$-UMST in Secs. 4 and 5, respectively.

### 4. Expected Path Optimality Condition

As introduced in Sec. 3.1, the path optimality condition (Theorem 3) is a necessary and sufficient condition of classical minimum spanning tree, providing a useful property for solving the MST problem. Following from this idea, in this section, we give two equivalent definitions of the expected UMST as well as the expected path optimality condition. After that, a method to find the expected UMST is given as well.

First of all, based on the linearity of expected value operator of uncertain variables (Theorem 2), we obtain the first equivalent definition as follows.

**Theorem 4.** (Equivalent Definition I of Expected UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is an uncertain expected minimum
spanning tree if and only if
\[
\sum_{e_i \in T^0} E[\xi_i] \leq \sum_{e_j \in T} E[\xi_j] \tag{20}
\]
holds for any spanning tree $T$, where $E[\xi_i]$ are the expected values of uncertain weights $\xi_i$, $i = 1, 2, \ldots, m$, respectively.

**Proof.** Since $\xi_i$, $i = 1, 2, \ldots, m$, are independent uncertain variables, following from Theorem 2, we have
\[
E[W(T, \xi)] = E \left[ \sum_{e_j \in T} \xi_j \right] = \sum_{e_j \in T} E[\xi_j]. \tag{21}
\]
Similarly,
\[
E[W(T^0, \xi)] = \sum_{e_i \in T^0} E[\xi_i]. \tag{22}
\]
Therefore, we get that $E[W(T^0, \xi)] \leq E[W(T, \xi)]$ holds if and only if $\sum_{e_i \in T^0} E[\xi_i] \leq \sum_{e_j \in T} E[\xi_j]$ is true. According to the definition of expected UMST (Definition 7), this theorem holds.

According to the path optimality condition for classical MST, a similar path optimality condition for the expected UMST can be obtained as follows.

**Theorem 5.** (Equivalent Definition II of Expected UMST, Expected Path Optimality Condition) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is an uncertain expected minimum spanning tree if and only if
\[
E[\xi_i] - E[\xi_j] \leq 0, \quad e_j \in E\backslash T^0, e_i \in P_j \tag{23}
\]
where $E\backslash T^0$ is the set of non-tree edges, and $P_j$ is the tree path of non-tree edge $e_j$.

**Proof.** Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, we can construct a corresponding deterministic graph $\tilde{G} = (V, E, E[\xi])$ consisting of the same vertex set $V$ and edge set $E$ with the uncertain graph $\tilde{G}$, as well as the deterministic edge weight vector $E[\xi] = (E[\xi_1], E[\xi_2], \ldots, E[\xi_m])^T$.

If $T^0$ is the expected UMST of $\tilde{G}$, $\sum_{e_i \in T^0} E[\xi_i] \leq \sum_{e_j \in T} E[\xi_j]$ holds for any spanning tree $T$ according to Theorem 4. Then $T^0$ is also the MST of $\tilde{G}$. Following from Theorem 3, (23) is obtained.

On the other hand, if $T^0$ satisfies (23), according to Theorem 3, $T^0$ is the MST of $\tilde{G}$. Then we have $\sum_{e_i \in T^0} E[\xi_i] \leq \sum_{e_j \in T} E[\xi_j]$ holds for any spanning tree $T$. Following from Theorem 4, $T^0$ is the expected UMST of $\tilde{G}$.

It follows from the expected path optimality condition (Theorem 5) that the expected UMST problem can be transformed to a classical MST problem. Moreover,
from the proving process of Theorem 5, we find that the expected UMST of the uncertain graph \( \tilde{G} = (V, E, \xi) \) is just the MST of the corresponding deterministic graph \( \bar{G} = (V, E, E[\xi]) \). We summarize this conclusion as the following theorem.

**Theorem 6.** (Equivalent Classical MST of Expected UMST) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), a spanning tree \( T^0 \) is the uncertain expected minimum spanning tree of \( \tilde{G} \) if and only if \( T^0 \) is the minimum spanning tree of a deterministic graph \( \bar{G} = (V, E, E[\xi]) \), where the deterministic edge weight vector is \( E[\xi] = (E[\xi_1], E[\xi_2], \ldots, E[\xi_m])^T \). 

**Proof.** It follows immediately from the proving process of Theorem 5.

Theorem 6 provides an effective method to solve the expected UMST problem. According to it, if we intend to find the expected UMST of a given uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi \) are uncertain edge weights, we can turn to find the MST of its corresponding deterministic graph by employing an existing algorithm for the classical MST problem (e.g. Kruskal algorithm). The process is summarized as follows.

**Method 1: Solving the expected UMST problem via deterministic graph transformation**

1. **Step 1.** Calculate \( E[\xi_i], i = 1, 2, \ldots, m \), respectively, and then obtain a deterministic graph \( \bar{G} = (V, E, E[\xi]) \).
2. **Step 2.** Use the Kruskal algorithm to find the minimum spanning tree of \( \bar{G} \), denoted by \( T^0 \).
3. **Step 3.** Return \( T^0 \) as the uncertain expected minimum spanning tree of \( \tilde{G} \).

**5. \( \alpha \)-Path Optimality Condition**

In this section, let us consider the uncertain \( \alpha \)-minimum spanning tree problem. Based on the inverse uncertainty distributions of uncertain variables, we propose three equivalent definitions of \( \alpha \)-UMST together with the \( \alpha \)-path optimality condition. Moreover, we also provide a method to find the uncertain \( \alpha \)-minimum spanning tree.

It can be deduced easily from Theorem 1 that the addition operation is close overall uncertain variables which have regular uncertainty distributions. Consequently, due to the uncertain edge weights, the weight \( W(T, \xi) \) of a spanning tree \( T \) is also an uncertain variable with a regular uncertainty distribution. Furthermore, for the inverse uncertainty distribution of \( W(T, \xi) \), we can obtain the following result.

**Theorem 7.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi \) are with regular uncertainty distributions \( \Phi_i, i = 1, 2, \ldots, m \), respectively, for any spanning
tree $T$, the weight $W(T, \xi)$ is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}_T(\alpha) = \sum_{e_i \in T} \Phi^{-1}_i(\alpha), \quad \alpha \in [0, 1].$$  \hfill (25)

**Proof.** Since $W(T, \xi) = \sum_{e_i \in T} \xi_i$ and $\xi_i$, $i = 1, 2, \ldots, m$, are independent uncertain variables with regular uncertainty distributions, this theorem follows from Theorem 1 immediately.

By utilizing the inverse uncertainty distribution of the weight of a spanning tree, we can obtain an equivalent definition of $\alpha$-UMST as follows.

**Theorem 8.** (Equivalent Definition I of $\alpha$-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ where $\xi_i$ are with regular uncertainty distributions $\Phi_i$, $i = 1, 2, \ldots, m$, respectively, and a predetermined confidence level $\alpha \in (0, 1]$, a spanning tree $T^0$ is an uncertain $\alpha$-minimum spanning tree if and only if

$$\Phi^{-1}_{T^0}(\alpha) \leq \Phi^{-1}_T(\alpha)$$  \hfill (26)

holds for any spanning tree $T$, where $\Phi^{-1}_{T^0}$ and $\Phi^{-1}_T$ are the inverse uncertainty distributions of tree weights $W(T^0, \xi)$ and $W(T, \xi)$, respectively.

**Proof.** According to Theorem 7, it is clear that the tree weight $W(T, \xi)$ is an uncertain variable with regular uncertainty distribution. Consequently, following from the definition of uncertainty distribution (see Definition 3), we have

$$\min\{\omega \mid \mathcal{M}(W(T, \xi) \leq \omega) \geq \alpha\} = \min\{\omega \mid \Phi_T(\omega) \geq \alpha\} = \Phi^{-1}_T(\alpha).$$  \hfill (27)

Similarly, for the spanning tree $T^0$, we also have

$$\min\{\omega \mid \mathcal{M}(W(T^0, \xi) \leq \omega) \geq \alpha\} = \Phi^{-1}_{T^0}(\alpha).$$  \hfill (28)

Therefore, we get that (19) holds, which means $T^0$ is the $\alpha$-UMST, if and only if $\Phi^{-1}_{T^0}(\alpha) \leq \Phi^{-1}_T(\alpha)$ holds.

This equivalent definition (Theorem 8) means that for a given confidence level $\alpha$, the uncertain $\alpha$-minimum spanning tree $T^0$ is in fact the one with the minimal $\alpha$-fractile with respect to the uncertainty distributions of the tree weights. Figure 4 shows the graphical interpretation of the $\alpha$-UMST.

Based on Theorem 7 and Theorem 8, we can further get another equivalent definition of $\alpha$-UMST using the inverse uncertainty distributions of the edge weights.

**Theorem 9.** (Equivalent Definition II of $\alpha$-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ where $\xi_i$ are with regular uncertainty distributions $\Phi_i$, $i =$
Fig. 4. Uncertain α-minimum spanning tree.

1, 2, ..., m, respectively, and a predetermined confidence level $\alpha \in (0, 1]$, a spanning tree $T^0$ is an uncertain α-minimum spanning tree if and only if

$$\sum_{e_i \in T^0} \Phi_e^{-1}(\alpha) \leq \sum_{e_j \in T} \Phi_e^{-1}(\alpha)$$

holds for any spanning tree $T$.

**Proof.** According to Theorem 7, for the spanning tree $T^0$ and any given spanning tree $T$, we have

$$\Phi_e^{-1}(\alpha) = \sum_{e_i \in T^0} \Phi_e^{-1}(\alpha)$$

and

$$\Phi_e^{-1}(\alpha) = \sum_{e_j \in T} \Phi_e^{-1}(\alpha).$$

Then this theorem follows immediately from (30), (31) and Theorem 8.

Similarly, like the path optimality conditions for the classical MST and expected UMST, we can extend the path optimality condition to the α-UMST, called the α-path optimality condition, which is also an equivalent definition of the α-UMST.

**Theorem 10.** (Equivalent Definition III of α-UMST, α-Path Optimality Condition) Given a connected uncertain graph $G = (V, E, \xi)$ where $\xi_i$ are with regular uncertainty distributions $\Phi_i$, $i = 1, 2, ..., m$, respectively, and a predetermined confidence level $\alpha \in (0, 1]$, a spanning tree $T^0$ is an uncertain α-minimum spanning tree if and only if

$$\Phi_e^{-1}(\alpha) - \Phi_e^{-1}(\alpha) \leq 0, e_j \in E \setminus T^0, e_i \in P_j$$

where $E \setminus T^0$ is the set of non-tree edges, and $P_j$ is the tree path of non-tree edge $e_j$. 
Path Optimality Conditions for Minimum Spanning Tree Problem

Proof. Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, we can construct a corresponding deterministic graph $\bar{G} = (V, E, \Phi^{-1}_i(\alpha))$ consisting of the same vertex set $V$ and edge set $E$ with uncertain graph $\tilde{G}$, as well as the deterministic edge weight vector $\Phi^{-1}_i(\alpha) = (\Phi^{-1}_1(\alpha), \Phi^{-1}_2(\alpha), \ldots, \Phi^{-1}_m(\alpha))^T$.

If $T^0$ is the $\alpha$-UMST of $\tilde{G}$, $\sum_{e_i \in T^0} \Phi^{-1}_i(\alpha) \leq \sum_{e_i \in T} \Phi^{-1}_i(\alpha)$ holds for any spanning tree $T$ according to Theorem 9. Then $T^0$ is also the MST of $\bar{G}$. Following from Theorem 3, (32) is obtained.

On the other hand, if $T^0$ satisfies (32), according to Theorem 3, $T^0$ is the MST of $\bar{G}$. Then we have $\sum_{e_i \in T^0} \Phi^{-1}_i(\alpha) \leq \sum_{e_i \in T} \Phi^{-1}_i(\alpha)$ holds for any spanning tree $T$. Following from Theorem 9, $T^0$ is the $\alpha$-UMST of $\bar{G}$.

Just like the deterministic graph transformation for the expected UMST problem, the $\alpha$-UMST problem can also be transformed to a classical MST problem following from the $\alpha$-path optimality condition as follows.

Theorem 11. Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ where $\xi_i$ are with regular uncertainty distributions $\Phi_i$, $i = 1, 2, \ldots, m,$ respectively, and a predetermined confidence level $\alpha \in (0, 1]$, a spanning tree $T^0$ is the uncertain $\alpha$-minimum spanning tree of $\tilde{G}$ if and only if $T^0$ is the minimum spanning tree of a deterministic graph $\bar{G} = (V, E, \Phi^{-1}_i)$, where the deterministic edge weight vector is

$$\Phi^{-1}_i(\alpha) = (\Phi^{-1}_1(\alpha), \Phi^{-1}_2(\alpha), \ldots, \Phi^{-1}_m(\alpha))^T.$$  

(33)

Proof. It follows immediately from the proving process of Theorem 10.

Theorem 11 describes the relationship between the $\alpha$-UMST of an uncertain graph and its counterpart of a deterministic graph, which also provides an effective method to solve the $\alpha$-UMST problem. Similarly with Method 1, the process of finding the $\alpha$-UMST of a given uncertain graph $\tilde{G} = (V, E, \xi)$ with a predetermined confidence level $\alpha \in (0, 1]$, where $\xi_i$, $i = 1, 2, \ldots, m$, are uncertain variables with regular uncertainty distributions $\Phi_i$, respectively, by employing Kruskal algorithm is summarized as follows.

Method 2: Solving the $\alpha$-UMST problem via deterministic graph transformation

Step 1. Calculate $\Phi^{-1}_i(\alpha)$, $i = 1, 2, \ldots, m$, respectively, and then obtain a deterministic graph $\bar{G} = (V, E, \Phi^{-1}_i(\alpha))$.

Step 2. Use the Kruskal algorithm to find the minimum spanning tree of $\bar{G}$, denoted by $T^0$.

Step 3. Return $T^0$ as the uncertain $\alpha$-minimum spanning tree of $\tilde{G}$.

6. Uncertain Most Minimum Spanning Tree

In this section, we define a new type of uncertain minimum spanning tree with a predetermined weight supremum $x^*$, called uncertain most minimum spanning tree.
Then the relationship between this new notion and the $\alpha$-UMST is discussed.

**Definition 9.** (Most UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ and a predetermined weight supremum $x^*$, a spanning tree $T^0$ is called an uncertain most minimum spanning tree if

$$\Phi_{T^0}(x^*) \geq \Phi_T(x^*)$$

(34)

holds for any spanning tree $T$.

This definition means that for a given appropriate weight supremum $x^*$, the uncertain most minimum spanning tree $T^0$ is the one with the maximal chance that the tree weight is less than or equal to the predetermined supremum $x^*$. Figure 5 shows the graphical interpretation of most UMST.

![Fig. 5. Uncertain most minimum spanning tree.](image)

**Remark 1.** In Peng and Li, three types of the uncertain minimum spanning tree were proposed, two of which, i.e., expected UMST and $\alpha$-UMST, have been discussed in Secs. 4 and 5 respectively. The third one is named as the uncertain distribution minimum spanning tree, which can be simply described as:

*Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is called an uncertain distribution minimum spanning tree if

$$\Phi_{T^0}(x) \leq \Phi_T(x)$$

(35)

holds for any spanning tree $T$ and any $x \in \mathbb{R}$.

The uncertain distribution minimum spanning tree implies that the chance of $T^0$ with a weight less than or equal to $x$ should be always less than or equal to any other spanning tree. However, this description is just contrary to the idea of minimum spanning tree, as the UMST defined by (35) is with more chances to get bigger weights. It in fact defines a type of “maximum spanning tree”. Therefore, this definition is usually not appropriate.
In order to prove the relationship between the proposed most UMST and the α-UMST, we first present the following theorem.

**Theorem 12.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i, i = 1, 2, \ldots, m \), respectively, for the set of all the spanning trees \( \{T^1, T^2, \ldots, T^k\} \) in \( \tilde{G} \), the function
\[
g(\alpha) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha)
\]
is a continuous and strictly increasing function with respect to \( \alpha \in (0, 1] \).

**Proof.** For the given graph \( \tilde{G} = (V, E, \xi) \), there are at most \( C^m_{n-1} \) spanning trees. That is, the number of possible spanning trees is a limited and fixed value. Thus, we can denote the set of all the spanning trees by listing them as \( \{T^1, T^2, \ldots, T^k\} \).

It follows from Theorem 7 that for any spanning tree \( T_j \), the tree weight \( W(T_j, \xi) \) is also an uncertain variable with regular uncertainty distribution, denoted as \( \Phi_{T_j}(x) \). Then \( \Phi_{T_j}(x) \) and \( \Phi_{T_j}^{-1}(\alpha) \) are both continuous and strictly increasing on the domains \( \{x | 0 < \Phi_{T_j}(x) < 1\} \) and \( \{\alpha | 0 < \alpha < 1\} \), respectively.

For arbitrary \( \alpha \in (0, 1] \), there exists an \( \alpha \)-UMST with \( \alpha \)-minimum weight, i.e., \( \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha) \). Although it is possible that there are more than one \( \alpha \)-UMST, they must have the equivalent \( \alpha \)-minimum weight. In other words, the minimum tree weight at confidence level \( \alpha \) is unique. It follows that \( g(\alpha) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1] \), which means \( g(\alpha) \) is a function map of \( \alpha \).

Now, let us prove \( g(\alpha) \) is continuous and strictly increasing. According to the arithmetic rule of limitation of compound function, we have
\[
\lim_{\alpha \to \alpha^*} g(\alpha) = \lim_{\alpha \to \alpha^*} \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha) = \min_{1 \leq j \leq k} \lim_{\alpha \to \alpha^*} \Phi_{T_j}^{-1}(\alpha)
\]
\[
= \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha^*) = g(\alpha^*)
\]
for any \( \alpha^* \in (0, 1] \). That means \( g(\alpha) \) is continuous.

For arbitrary \( 0 < \alpha_1 < \alpha_2 \leq 1 \), we get
\[
g(\alpha_1) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha_1) \quad \text{and} \quad g(\alpha_2) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha_2).
\]
Since \( \Phi_{T_j}^{-1}(\alpha_1) < \Phi_{T_j}^{-1}(\alpha_2) \) for \( j = 1, 2, \ldots, k \), we have
\[
\min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha_1) < \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha_2).
\]
That is, \( g(\alpha_1) < g(\alpha_2) \). Hence \( g(\alpha) \) is strictly increasing with respect to \( \alpha \in (0, 1] \).

**Remark 2.** It is clear that an appropriate weight supremum \( x^* \) for the most UMST falls in the interval \( g(0) < x^* \leq g(1) \), where
\[
g(0) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(0) = \min_{1 \leq j \leq k} \lim_{\alpha \to 0^+} \Phi_{T_j}^{-1}(\alpha)
\]
and

\[ g(1) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(1) = \min_{1 \leq j \leq k} \lim_{\alpha \to 1^-} \Phi_{T_j}^{-1}(\alpha). \]  

(41)

In other words, if \( x^* \) falls out of this domain, the concept of most UMST is not well defined. This is also make sense in practice. Since the tree weights distribute in some corresponding intervals, it makes the notion of most UMST useful only with some appropriate weight supremums.

Based upon Theorem 12, we have the following conclusion concerning the relationship between the most UMST and the \( \alpha \)-UMST.

**Theorem 13.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \), \( i = 1, 2, \ldots, m \), respectively, and a predetermined weight supremum \( x^* \in [g(0), g(1)] \), a spanning tree \( T^0 \) is the uncertain most minimum spanning tree of \( \tilde{G} \) if and only if \( T^0 \) is the uncertain \( \alpha^* \)-minimum spanning tree of \( \tilde{G} \) where \( \alpha^* = g^{-1}(x^*) \).

**Proof.** For the \( \alpha \)-minimum weight \( g(\alpha) \), we know it is a continuous and strictly increasing function with respect to \( \alpha \in (0, 1) \) from Theorem 12. Consequently, it is easy to get the range of \( g(\alpha) \) as \( g(0) < g(\alpha) \leq g(1) \).

Since \( g(\alpha) \) is a continuous and strictly increasing function with respect to \( \alpha \in (0, 1] \), then for any given \( x \in [g(0), g(1)] \), there exists \( \alpha \in (0, 1] \) such that \( \alpha = g^{-1}(x) \).

If \( T^0 \) is the most UMST with respect to the predetermined supremum \( x^* \), we have

\[ \Phi_{T^0}(x^*) = \max_{1 \leq j \leq k} \Phi_{T_j}(x^*). \]  

(42)

Then denote

\[ \alpha_{T^0} = \Phi_{T^0}(x^*) \text{ and } \alpha_{T_j} = \Phi_{T_j}(x^*). \]  

(43)

We have

\[ \alpha_{T^0} = \max_{1 \leq j \leq k} \alpha_{T_j} \]  

(44)

and

\[ \Phi_{T_j}^{-1}(\alpha_{T^0}) = \Phi_{T_j}^{-1}(\alpha_{T_j}) = x^*. \]  

(45)

Since \( \Phi_{T_j}^{-1} \) is strictly increasing, we have that for any spanning tree \( T_j \),

\[ \Phi_{T_j}^{-1}(\alpha_{T^0}) \geq \Phi_{T_j}^{-1}(\alpha_{T_j}) = \Phi_{T_j}^{-1}(\alpha_{T^0}) \]  

(46)

which means \( T^0 \) is the \( \alpha_{T^0} \)-UMST.

According to the definition of \( g(\alpha) \), we also have \( g(\alpha_{T^0}) = \Phi_{T^0}^{-1}(\alpha_{T^0}) \). Since \( \alpha_{T^0} = \Phi_{T^0}(x^*) \), we get \( g(\alpha_{T^0}) = x^* \), and thus \( \alpha^* = g^{-1}(x^*) = \alpha_{T^0} \). That means \( T^0 \) is the \( \alpha^* \)-UMST of \( \tilde{G} \).
Path Optimality Conditions for Minimum Spanning Tree Problem

On the other hand, if $T^0$ is the $\alpha^*$-UMST with $\alpha^* = g^{-1}(x^*)$, then we have
\[
x^* = \min_{1 \leq j \leq k} \Phi_{T^j}^{-1}(\alpha^*) = \Phi_{T^0}^{-1}(\alpha^*)
\]
which may deduce that $\alpha^* = \Phi_{T^0}(x^*)$ and
\[
\Phi_{T^0}(x^*) = \max_{1 \leq j \leq k} \Phi_{T^j}(x^*) \geq \Phi_{T^j}(x^*).
\]
That means $T^0$ is the most UMST of $\tilde{G}$.

So far, we have discussed some important properties of the three types of UMST, i.e., the expected UMST, $\alpha$-UMST, and most UMST. According to Theorems 6, 11 and 13, all the three types of UMST can be transformed to their equivalent counterparts in the corresponding deterministic graphs, which means the UMST problem can be handled within the framework of classical MST problem and requires no particular solving methods. We just need to employ a classical algorithm (for instance, the Kruskal algorithm) to find the MST of a deterministic graph. Obviously, finding a UMST by taking this approach has a low computation complexity, which is the same as the employed algorithm for classical MST problem (taking the Kruskal algorithm for example, it is $O(m \log n)$). With the aid of some well-developed optimization software packages, such as LINDO or MATLAB, the problem may be solved to optimality for scenarios of moderate size or even large size.

7. Numerical Examples

In this section, we give some numerical examples of the uncertain minimum spanning tree problems to illustrate the conclusions presented above. A connected graph $\tilde{G}$ as shown in Fig. 6 consists of 6 vertices and 8 edges. The edge weights are uncertain variables, denoted as $\xi_i$ ($i = 1, 2, \ldots, 8$).

![Fig. 6. Uncertain graph for numerical examples.](image)

Suppose that $\xi_1, \xi_3, \xi_5$, and $\xi_7$ are independent linear uncertain variables, and $\xi_2, \xi_4, \xi_6$, and $\xi_8$ are independent zigzag uncertain variables, as listed in Table 1. The inverse uncertainty distributions of linear and zigzag uncertain variables can
be calculated following from (9) and (10), respectively. In addition, the expected values of a linear uncertain variable \( \xi \sim L(a, b) \) and a zigzag uncertain variable \( \xi \sim Z(a, b, c) \) are

\[
E[\xi] = \frac{a + b}{2} \quad (49)
\]

and

\[
E[\xi] = \frac{a + 2b + c}{4} \quad (50)
\]

respectively, according to Definition 5.

As a result, we can obtain the expected values \( E[\xi_i] \) as well as \( \Phi^{-1}_i(\alpha) \) for a given confidence level \( \alpha = 0.8 \), \( i = 1, 2, \ldots, 8 \), which are also shown in Table 1.

<table>
<thead>
<tr>
<th>Edge ( i )</th>
<th>( \xi_i )</th>
<th>( E[\xi_i] )</th>
<th>( \Phi^{-1}_i(0.8) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( L(20, 30) )</td>
<td>25</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>( Z(12, 16, 40) )</td>
<td>21</td>
<td>30.4</td>
</tr>
<tr>
<td>3</td>
<td>( L(20, 32) )</td>
<td>26</td>
<td>29.6</td>
</tr>
<tr>
<td>4</td>
<td>( Z(11, 23, 29) )</td>
<td>21.5</td>
<td>26.6</td>
</tr>
<tr>
<td>5</td>
<td>( L(18, 26) )</td>
<td>22</td>
<td>24.4</td>
</tr>
<tr>
<td>6</td>
<td>( Z(14, 20, 22) )</td>
<td>19</td>
<td>21.2</td>
</tr>
<tr>
<td>7</td>
<td>( L(11, 27) )</td>
<td>19</td>
<td>23.8</td>
</tr>
<tr>
<td>8</td>
<td>( Z(10, 12, 20) )</td>
<td>13.5</td>
<td>16.8</td>
</tr>
</tbody>
</table>

**Example 2.** In order to find the expected UMST of the graph \( \tilde{G} \) in Fig. 6, we can solve the expected UMST problem via deterministic graph transformation as Method 1 shows. The corresponding deterministic graph is shown in Fig. 7. Then we can find out the MST of Fig. 7 as shown in Fig. 8, which is also the expected UMST of Fig. 6. The expected UMST consists of edges \( e_2 \), \( e_4 \), \( e_6 \), \( e_7 \), and \( e_8 \) with the expected minimum weight

\[
\]

**Example 3.** In order to find the \( \alpha \)-UMST of Fig. 6, we can also solve the \( \alpha \)-UMST problem via deterministic graph transformation as Method 2 shows. For a given confidence level \( \alpha = 0.8 \), the corresponding deterministic graph with edge vector \( \Phi^{-1}_\alpha(0.8) \) is shown in Fig. 9. Then we can find out the MST of Fig. 9 as shown in Fig. 10, which is also the 0.8-UMST of Fig. 6. The 0.8-UMST consists of edges \( e_3 \), \( e_5 \), \( e_6 \), \( e_7 \), and \( e_8 \) with the 0.8-minimum weight

\[
W_{0.8}(T^0, \xi) = \Phi^{-1}_3(0.8) + \Phi^{-1}_5(0.8) + \Phi^{-1}_6(0.8) + \Phi^{-1}_7(0.8) + \Phi^{-1}_8(0.8) = 29.6 + 24.4 + 21.2 + 23.8 + 16.8 = 115.8. \quad (52)
\]
Comparing Figs. 8 and 10, it is easy to conclude that different definitions of uncertain minimum spanning tree may lead to different optimal solutions.

For the \( \alpha \)-UMST problem, the predetermined confidence level \( \alpha \) may play an important impact on the found UMST. The numerical example is further considered with different confidence levels. Table 2 together with their corresponding optimal solutions, which demonstrates the changes of the found \( \alpha \)-UMST with respect to different confidence level. Figures 11–13 present the \( \alpha \)-UMST when \( \alpha \) is 0.3, 0.6, and 0.9, respectively. The results show that the confidence level has an important effect on the found \( \alpha \)-UMST, and the \( \alpha \)-minimum weight increases while the confidence level is increasing.

Example 4. For the most UMST problem, it is clear that it can be transformed to an equivalent \( \alpha \)-UMST problem following from Theorem 13. For example, given a weight supremum \( x^* = 121.4 \), from Table 2, we have \( g^{-1}(121.4) = \Phi^{-1}(121.4) = 0.9 \). Then we get that the most UMST of Fig. 6 with \( x^* = 121.4 \) is just its 0.9-UMST, as shown in Fig. 13.
Table 2. Results with different confidence levels.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$T^0$</th>
<th>$W_\alpha(T, \xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$e_2, e_4, e_6, e_7, e_8$</td>
<td>64.4</td>
</tr>
<tr>
<td>0.2</td>
<td>$e_2, e_4, e_6, e_7, e_8$</td>
<td>70.8</td>
</tr>
<tr>
<td>0.3</td>
<td>$e_2, e_4, e_6, e_7, e_8$</td>
<td>77.2</td>
</tr>
<tr>
<td>0.4</td>
<td>$e_2, e_4, e_6, e_7, e_8$</td>
<td>83.6</td>
</tr>
<tr>
<td>0.5</td>
<td>$e_2, e_5, e_6, e_7, e_8$</td>
<td>89</td>
</tr>
<tr>
<td>0.6</td>
<td>$e_3, e_5, e_6, e_7, e_8$</td>
<td>98.2</td>
</tr>
<tr>
<td>0.7</td>
<td>$e_3, e_5, e_6, e_7, e_8$</td>
<td>107.4</td>
</tr>
<tr>
<td>0.8</td>
<td>$e_3, e_5, e_6, e_7, e_8$</td>
<td>115.8</td>
</tr>
<tr>
<td>0.9</td>
<td>$e_3, e_5, e_6, e_7, e_8$</td>
<td>121.4</td>
</tr>
</tbody>
</table>

More generally, for any $\alpha \in (0, 1]$, we have $58 < g(\alpha) = W_\alpha(T^0, \xi) \leq 127$, following from (40) and (41). Figure 14 shows the $\alpha$-minimum weights of the spanning trees, i.e., $g(\alpha)$, with respect to different confidence levels. Consequently, for any given weight supremum $58 < x^* \leq 127$, we can get $g^{-1}(x^*)$ according to Fig. 14, which means the most UMST problem can always be transformed to its equivalent $\alpha^*$-UMST problem, where $\alpha^* = g^{-1}(x^*)$.

Remark 1. As mentioned in Remark 1, the definition of uncertain distribution minimum spanning tree in Peng and Li\(^\text{18}\) is not appropriate. If we modify the formula $\Phi_{T^0}(x) \leq \Phi_T(x)$, i.e., (35), in the definition of uncertain distribution minimum spanning tree to $\Phi_{T^0}(x) \geq \Phi_T(x)$, which means the weight of $T^0$ has more chance of being less than or equal to $x$ than any other spanning tree, then the definition would be more appropriate. In other words, the uncertain distribution minimum spanning tree defined by $\Phi_{T^0}(x) \geq \Phi_T(x)$ means that $T^0$ must be the optimal solution of the most UMST problem for any predetermined weight supremum $x^*$ or the
optimal solution of the \( \alpha \)-UMST problem for any confidence level \( \alpha \in (0, 1) \). In this case, this definition may lead to some uncertain minimum spanning tree problems with no optimal solution as illustrated in Examples 3 and 4.

8. Conclusion

Since the applications of the minimum spanning tree problem encountered in practice usually involve some uncertain issues, the edge weights cannot be explicitly determined. In this paper, we discussed the minimum spanning tree problem where edge weights are uncertain variables. It is shown that the notions of the uncertain expected minimum spanning tree and the uncertain \( \alpha \)-minimum spanning tree can be characterized by a set of constraints on non-tree edges and their corresponding tree paths. By this characterization, referred to as the path optimality conditions, we proved that the two types of UMST have equivalent counterparts in their corresponding deterministic graphs. In other words, both the expected UMST problem and the \( \alpha \)-UMST problem can be transformed to the classical MST problems and then be solved efficiently by taking the advantage of existing algorithms or some well-developed software packages. Furthermore, we also presented a new definition of uncertain minimum spanning tree, i.e., the uncertain most minimum spanning tree, and revealed the equivalent relationship between the \( \alpha \)-UMST and the most UMST. In conclusion, the solving processes of these three types of UMST problems are summarized in Fig. 15.

![Fig. 15. Solving processes of three types of UMST problems.](image-url)
Acknowledgements

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References


