Shortest path problem of uncertain random network

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\begin{abstract}
The shortest path problem is one of the most fundamental problems in network optimization. This paper is concerned with shortest path problems in non-deterministic environment, in which some arcs have stochastic lengths and meanwhile some have uncertain lengths. In order to deal with path problems in such network, this paper introduces the chance theory and uses uncertain random variable to describe non-deterministic lengths, based on which two types of shortest path in uncertain random networks are defined. To obtain the shortest paths, an algorithm derived from the Dijkstra Algorithm is proposed. Finally, numerical examples are given to illustrate the effectiveness of the algorithms.
\end{abstract}

\section{1. Introduction}

As one of the fundamental problems in network optimization, the shortest path problem concentrates on finding a path with minimum distance, time or cost from the source to the destination. Some other optimization problems, such as transportation, communications and supply chain management, can be considered as special cases of the shortest path problem. Since the late 1950s, the shortest path problem and problems stemming from it have been widely studied, and some successful algorithms have been proposed, such as Bellman (1958), Dijkstra (1959), Dreyfus (1969), and Floyd (1962).

In traditional networks, the length(weight) of each arc is deterministic. However, because of failure, maintenance, or other reasons, arc lengths are non-deterministic in many situations. Some researchers believed that these non-deterministic phenomena conform to randomness, so they introduced probability theory into network optimization problems and used random variables to describe the non-deterministic lengths. Random network was first investigated by Frank and Hakimi (1965) in 1965 for modeling communication networks with random capacities. From then on, random networks have been well developed and widely applied. Many researchers have done lots of work on random shortest path problems, such as Frank (1969), Nie and Wu (2009), Chen, Lam, Sumalee, and Li (2012), and Zockaie, Nie, and Mahmassani (2014). In these literature, the weights of arcs were regarded as random variables, and corresponding models were studied. However, such probability-based methods can effectively work only when probability distribution functions of non-deterministic phenomena are exactly obtained. That is to say, if there are not enough observational data to support the non-deterministic phenomena, the probability distribution functions estimated via statistical methods will not be close to real situations, which indicates that it is not suitable to employ random variables to model non-deterministic phenomena in these cases.

In the cases that there is little or no observational data for non-deterministic phenomena, a feasible and economic way is to estimate the data by experts based on their subjective information and experiences. As a matter of course, additivity and some other specific properties of probability calculus fall away when there are only expert empirical data, since human factors are so imprecise and there is almost no rigorous additive property can be discovered (Kóczy, 1992). To describe expert data in arc lengths, uncertainty theory (Liu, 2007, 2010) was introduced into network optimization problems and uncertain variables were employed to model arc lengths. Uncertain network was first explored by Liu (2009) for modeling project scheduling problem with uncertain duration times. In 2011, Gao (2011) deduced the uncertainty distribution function of the shortest path length, and studied the $\alpha$-shortest path and the most measure shortest path in uncertain networks. In 2014, Han, Peng, and Wang (2014) studied maximum flow problems in networks with uncertain capacities, and designed a 99-algorithm to calculate the uncertainty distribution function of maximum flow. In 2015, Gao, Yang, Li, and Kar (2015) and Gao and Qin (in press) investigated uncertain graphs, and proposed efficient algorithms to calculate the distribution function of the diameter and the edge-connectivity of an uncertain graph. Besides, the
uncertain minimum cost flow problem was investigated by Ding (2014), and the Chinese postman problem was studied by Zhang and Peng (2012).

In some real cases, uncertainty and randomness simultaneously appear in a system. Specifically, for some non-deterministic phenomena, we have enough observational data to obtain their probability distribution functions; while for others, we can only estimate them by expert data. In order to deal with this kind of systems, Liu (2013) proposed chance theory in 2013. Further, Liu (2014) introduced the chance theory into networks, which involves both random arc lengths and uncertain arc lengths. In the paper, Liu proposed a concept of uncertain random network, in which the lengths of some arcs are random variables and other lengths are uncertain variables. Furthermore, Liu studied the chance distribution function of minimum length from source node to destination node in uncertain random networks. Besides, models for maximum flow problems in uncertain random networks were constructed by Sheng and Gao (2014).

This paper will further study the shortest path problem within the framework of chance theory. The contributions of this paper are threefold. First, we propose an algorithm, which is derived from Dijkstra Algorithm, to calculate the chance distribution function of the minimum length from source node to destination node in an uncertain random network. Note that in Liu (2014), only a theoretical formula was proposed to calculate the chance distribution function of the minimum length, which is difficult to use in practice. The proposed algorithm in this paper numerically calculates the chance distribution function.

Second, two types of shortest paths in uncertain random networks are first proposed, which are simply named as type-I shortest path and type-II shortest path, respectively. In an uncertain random network, the length of a path is an uncertain random variable, instead of a constant. As a result, the traditional concept of shortest path is not suitable in uncertain random networks any more. The definitions of type-I and type-II shortest paths are both based on minimizing deviations between the chance distribution function of the length of a path and the chance distribution function of the minimum length from source node to destination node. The deviations between chance distribution functions are presented in Section 4 in detail.

Third, optimization models are respectively constructed to model the proposed types of shortest paths. The properties of the models are investigated, based on which an algorithm is developed to solve the model. The effectiveness of the algorithm is illustrated by several numerical examples.

It should be pointed out that in this paper, we don't distinguish risk-averse decision-makers and risk-prone users decision-makers, that is, we assume that all the decision-makers have the same perspective on uncertainty. Besides, all the uncertain lengths and random lengths are assumed to be independent throughout this paper. The remainder of this paper is organized as follows. In Section 2, some basic concepts of uncertainty theory and chance theory used throughout this paper are introduced. Section 3 proposes an algorithm to calculate the chance distribution function of the minimum length from the source node to the destination node in uncertain random networks. In Section 4, two types of shortest paths in uncertain random networks are proposed and investigated in details, and then an algorithm is designed to search the corresponding paths. In Section 5, a numerical example is given to illustrate the proposed model and algorithm. Section 6 gives a brief summary to this paper.

2. Preliminaries

Recall that uncertainty theory and probability theory are two different mathematical tools to describe and model non-deterministic phenomena. Specifically, if we have enough historical or experimental data for the non-deterministic phenomena, then these phenomena can be described by random variables; while if there are only expert empirical data to estimate the non-deterministic phenomena, it is better to use uncertain variables to describe these phenomena.

In this section, we first review some concepts of uncertainty theory, including uncertain measure, uncertain variable and operational law. Then we introduce some useful definitions and properties about chance theory, such as uncertain random variable and chance distribution function.

2.1. Uncertainty theory

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is designated a number $\mathcal{M}(\Lambda)$. Liu (2007) proposed four axioms to ensure that the set function $\mathcal{M}(\Lambda)$ satisfy certain mathematical properties.

**Axiom 1** (Normality). $\mathcal{M}(\Gamma) = 1$.

**Axiom 2** (Duality). $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event $\Lambda$.

**Axiom 3** (Subadditivity). For every countable sequence of events $(\Lambda_i)$, we have

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).$$

**Axiom 4** (Product Axiom). Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ be uncertainty spaces for $i = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left(\prod_{i=1}^{\infty} \Lambda_i \right) = \prod_{i=1}^{\infty} \mathcal{M}_i(\Lambda_i),$$

where $\Lambda_1$ is the maximum value operator, and $\Lambda_i$ are arbitrarily chosen events from $\mathcal{L}_i$ for $i = 1, 2, \ldots$, respectively.

Note that the first three axioms of uncertainty theory also hold in probability theory, while the product axiom of uncertainty theory is totally different from that of probability theory, which leads to the difference of these two theories.

To describe a quantity with uncertainty, uncertain variable is defined analogous to the definition of random variable.

**Definition 1** Liu (2007). An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Let $x \in \mathbb{R}$ and $B = (-\infty, x)$. Then $\Phi(x) = \mathcal{M}(\{\xi \in B\} = \mathcal{M}(\xi < x)$ is called the uncertainty distribution function of $\xi$. The inverse function $\Phi^{-1}$ is called the inverse uncertainty distribution function of $\xi$. If it exists and is unique for each $x \in (0, 1)$, inverse uncertainty distribution function plays a crucial role in operations of independent uncertain variables.

The following theorem presents the operational law of independent uncertain variables, which gives the uncertainty distribution function of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$.

**Theorem 1** Liu (2010). Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distribution functions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is strictly increasing function, then
\[ \zeta = f(\xi_1, \xi_2, \ldots, \xi_n) \]
is an uncertain variable with uncertain distribution function
\[ \Phi(x) = \sup_{\Phi(x) = \min} \Phi(x). \]

Using inverse uncertain distribution function, the operation law can be formulated as

**Theorem 2 Liu (2010).** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with uncertain distribution functions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(\xi_1, \xi_2, \ldots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \ldots, \xi_n \) and strictly decreasing with respect to \( \xi_{n+1}, \xi_{n+2}, \ldots, \xi_m \), then \( \zeta = f(\xi_1, \xi_2, \ldots, \xi_n) \) is an uncertain variable with an inverse uncertain distribution function
\[ \Phi^{-1}(x) = f(\Phi_1^{-1}(x), \ldots, \Phi_n^{-1}(x), \Phi_{n+1}^{-1}(1-x), \ldots, \Phi_m^{-1}(1-x)). \]

Obviously, the inverse function of \( \Phi^{-1} \) is the uncertain distribution function of \( \zeta \), i.e., \( \Phi \).

### 2.2. Chance theory

The chance space is the product of \( (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P}) \), in which \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is an uncertainty space and \( (\Omega, \mathcal{A}, \mathcal{P}) \) is a probability space.

**Definition 2 Liu (2013).** Let \( (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P}) \) be a chance space, and let \( \Theta \in \mathcal{L} \times \mathcal{A} \) be an uncertain random event. Then the chance measure of \( \Theta \) is defined as
\[ Ch(\Theta) = \int_0^1 Pr(\omega \in \Omega, \mathcal{M} : \gamma \in \Gamma, (\gamma, \omega) \in \Theta) \, dr. \]

Liu (2013) proved that a chance measure satisfies normality, duality, and monotonicity properties, that is (i) \( Ch(\Gamma \times \Omega) = 1 \); (ii) \( Ch(\Theta) + Ch(\Theta^c) = 1 \) for any event \( \Theta \); (iii) \( Ch(\Theta_1) \leq Ch(\Theta_2) \) for any real number set \( \Theta_1 \subset \Theta_2 \). Besides, Hou (2014) proved the subadditivity of chance measure, that is,
\[ Ch\left( \bigcup_{i=1}^{n} \Theta_i \right) \leq \sum_{i=1}^{n} Ch(\Theta_i) \]
for a sequence of events \( \Theta_1, \Theta_2, \ldots \).

**Definition 3 Liu (2013).** An uncertain random variable is a function \( \zeta \) from a chance space \( (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P}) \) to the set of real numbers such that \( \{ \zeta \in B \} \) is an event in \( \mathcal{L} \times \mathcal{A} \) for any Borel set \( B \) of real numbers.

**Definition 4 Liu (2013).** Let \( \zeta \) be an uncertain random variable. Then its chance distribution function is defined by
\[ \Phi(x) = Ch(\zeta \leq x) \]
for any \( x \in \mathbb{R} \).

The chance distribution function of a random variable is just its probability distribution function, and the chance distribution function of an uncertain variable is just its uncertainty distribution function. The following theorem presents the operational law of uncertain random variables.

**Theorem 3 Liu (2013).** Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent uncertain random variables with probability distribution functions \( G_1, G_2, \ldots, G_m \), respectively, and let \( \tau_1, \tau_2, \ldots, \tau_n \) be uncertain variables. Then the uncertain random variable
\[ \zeta = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \]
has a chance distribution function
\[ \Phi(x) = \int_{\mathbb{R}^m} F(x,y_1,\ldots,y_m) \, dG_1(y_1) \ldots dG_m(y_m) \]
where \( F(x,y_1,\ldots,y_m) \) is the uncertainty distribution function of uncertain variable \( f(y_1,y_2,\ldots,y_m,\tau_1,\tau_2,\ldots,\tau_n) \) for any real numbers \( y_1,y_2,\ldots,y_n \).

### 3. The chance distribution of the minimum length

In this section, we introduce the definition of uncertain random networks and the chance distribution function of the minimum length from source node to destination node. An algorithm for calculating the chance distribution function is proposed. First, some assumptions are listed as follows.

1. There is only one source node and only one destination node.
2. The length of each arc \( (i,j) \) is finite.
3. The length of each arc \( (i,j) \) is either a positive uncertain variable or a positive random variables.
4. All the uncertain variables and the random variables are independent.

Assume that the node set of the uncertain random network is \( N = \{1,2,\ldots,n\} \), the uncertain arc set is \( U \) and the random arc set is \( R \), i.e.,
\[ U = \{(i,j) | arc (i,j) has uncertain length\}, \]
\[ R = \{(i,j) | arc (i,j) has random length\}. \]

In this paper, all deterministic arcs are regarded as special uncertain arcs. Let \( \zeta_{ij} \) denote the length of arc \( (i,j) \), \( (i,j) \in U \cup R \). Then \( \zeta_{ij} \) are uncertain variables if \( (i,j) \in U \), and \( \zeta_{ij} \) are random variables if \( (i,j) \in R \). Since the length of each arc is assumed to be finite, we have \( a_{ij} \leq \zeta_{ij} \leq b_{ij} \), where \( a_{ij} \) and \( b_{ij} \) are the lower bound and upper bound of \( \zeta_{ij} \) respectively. For simplicity, we define \( \xi = \{\zeta_{ij}\} \), for \( (i,j) \in U \cup R \).

**Definition 5 Liu (2014).** Assume \( N \) is the set of nodes, \( U \) is the set of uncertain arcs, \( R \) is the set of random arcs, and \( \xi \) is the set of uncertain and random lengths. Then the quartette \( (N,U,R,\xi) \) is called an uncertain random network.

The uncertain random network becomes a random network (Frank & Hakimi, 1965) if all arc lengths are random variables; and becomes an uncertain network (Liu, 2010) if all arc lengths are uncertain variables.

In a deterministic network \( (N,A,W) \), where \( A \) is the set of arcs and \( W = \{w_{ij} | (i,j) \in A \} \) is the set of arc lengths, the shortest path from the source node to the destination node can be easily found by the Dijkstra Algorithm (Dijkstra, 1959). Obviously, the length of the shortest path, i.e., \( f(W) \), is an increasing function of length \( w_{ij} \) for all \( (i,j) \in A \). Actually, it is easy to verify that
\[ f(W_1) \leq f(W_2), \]
where \( w_1 = \{w_{ij} | (i,j) \in A \} \), \( w_2 = \{w_{ij} | (i,j) \in A \} \), and \( w_1 \leq w_2 \), for all \( (i,j) \in A \).

For an uncertain random network \( (N,U,R,\xi) \), since the length of each arc is non-deterministic, the concept of the shortest path is different from the deterministic network. With the length taking different values, the shortest path from the source node to the destination node varies, which indicates that we may not find a path that is always the shortest. However, the chance distribution function of the minimum length can be obtained by the following method.
Theorem 4 Liu (2014). For uncertain random network \((N, U, R, \xi)\), assume the uncertain lengths \(z_q\) has uncertainty distribution function \(Y_q\) for any \((i, j) \in U\), and the random lengths \(w_q\) has probability distribution function \(G_q\) for any \((i, j) \in R\), respectively. Then the chance distribution function of the minimum length from the source node to the destination node is

\[
\Phi(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} F(x; y_{ij}, i, j \in R) \prod_{(i, j) \in R} dG_q(y_q),
\]

where \(F(x; y_{ij}, i, j \in R)\) is determined by its inverse uncertainty distribution function

\[
F^{-1}(x; y_{ij}, i, j \in R) = f(c_q, i, j \in U \cup R),
\]

and \(f\) is the length of shortest path of the deterministic network \((N, U \cup R, \Psi)\), where \(\Psi = \{c_q\}\). Obviously, \(f\) can be calculated by the Dijkstra Algorithm.

The following theorem is obvious.

Theorem 5. Let \(P\) be a path from the source node to the destination node of an uncertain random network \((N, U, R, \xi)\) and let \(\Psi(x)\) be the chance distribution function of the length of path \(P\). Then we always have

\[
\Psi(x) \leq \Phi(x)
\]

where \(\Phi(x)\) is the chance distribution function of the minimum length from the source node to the destination node.

Note that formula (1) is only a theoretical formula to calculate \(\Phi(x)\), which is difficult to use. In order to calculate the chance distribution function of the minimum length from source node to destination node in an uncertain random network, we propose the following algorithm:

Algorithm 1.

Step 1. Discretize the length of each random arc. To be more precise, for random arc \((i, j) \in R\), give a partition \(\Pi_i\) on interval \([a_i, b_i]\) with step \(\Delta_i = 0.01\). Then, random length \(z_q\) takes value in \(\Pi_i\).

Step 2. Denote \(A\) as a partition on interval \((0, 1)\). For all \(y_q \in \Pi_i\) and \(x \in A\), calculate \(F^{-1}(x; y_{ij}, i, j \in R)\) according to formulas (2) and (3), then, obtain the discrete form of \(F(x; y_{ij}, i, j \in R)\).

Step 3. Obtain the uncertainty distribution function of \(F(x; y_{ij}, i, j \in R)\) from its discrete form via linear interpolation.

Step 4. Input \(F(x; y_{ij}, i, j \in R)\) into formula (1), then obtain the chance distribution function \(\Phi(x)\).

Obviously, Algorithm 1 is only an approximate method. However, the smaller the steps of the partitions are, the more precise chance distribution function Algorithm 1 obtain. To illustrate Algorithm 1, we present the following example.

Example 1. Uncertain random network \((N, U, R, \xi)\) has 4 nodes and 4 arcs, which is shown in Fig. 1. Assume that the uncertain lengths \(\tau_{24}, \tau_{34}\) have regular uncertainty distribution functions \(Y_{24}, Y_{34}\), and the random lengths \(\xi_{12}, \xi_{13}\) have probability distribution functions \(G_{12}, G_{13}\), respectively. Then the minimum length from node 1 to node 4 is an uncertain random variable, that is, \((\xi_{12} + \tau_{24}) \land (\xi_{13} + \tau_{34})\). According to Theorem 4, the chance distribution function of the minimum length can be obtained by

\[
\Phi(x) = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} F(x; y_{12}, y_{13}) d\tau_{12} d\tau_{13},
\]

where \(F(x; y_{12}, y_{13})\) is the uncertainty distribution function of uncertain variable \((\xi_{12} + \tau_{24}) \land (\xi_{13} + \tau_{34})\). Given \(x \in (0, 1)\), the inverse uncertainty distribution function of \(F(x; y_{12}, y_{13})\) can be calculated by

\[
F^{-1}(x; y_{12}, y_{13}) = (y_{12} + Y_{24}^{-1}(x)) \land (y_{13} + Y_{34}^{-1}(x)).
\]

For simplicity, assume that the uncertain lengths follow the linear distribution function, i.e., \(\tau_{24} \sim U(2.5, 3.4)\), \(\tau_{34} \sim U(3, 4)\), and the random lengths follow the uniform distribution function, i.e., \(\xi_{12} \sim U(1, 3)\), \(\xi_{13} \sim U(2, 3)\). For given \(x \in (0, 1)\), the inverse uncertainty distribution function of \(\tau_{24}\) and \(\tau_{34}\) are respectively

\[
\begin{align*}
Y_{24}^{-1}(x) &= 2 + 3x, \quad Y_{34}^{-1}(x) = 3 + x.
\end{align*}
\]

The probability distribution function of \(\xi_{12}\) and \(\xi_{13}\) are respectively

\[
\begin{align*}
G_{12}(y_{12}) &= \begin{cases} 
0, & \text{if } y_{12} < 0 \\
\frac{y_{12} - 1}{2}, & \text{if } 0 \leq y_{12} < 3 \\
1, & \text{if } y_{12} \geq 3.
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
G_{13}(y_{13}) &= \begin{cases} 
0, & \text{if } y_{13} < 2 \\
y_{13} - 2, & \text{if } 2 \leq y_{13} < 3 \\
1, & \text{if } y_{13} \geq 3.
\end{cases}
\end{align*}
\]

Then, the chance distribution function of the minimum length can be reformulated as

\[
\Phi(x) = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} F(x; y_{12}, y_{13}) d\tau_{12} d\tau_{13},
\]

where \(F(x; y_{12}, y_{13})\) is determined by the inverse uncertainty distribution function

\[
F^{-1}(x; y_{12}, y_{13}) = (y_{12} + (2 + 3x)) \land (y_{13} + (3 + x)).
\]

for each given \(x \in (0, 1)\).

Recall that formula (4) is only a theoretical formula. Next, we use Algorithm 1 to obtain the chance distribution function.

Given a partition \(\Pi_{13}\) on interval \([1, 3]\) with step 0.2, and let random variable \(\xi_{12}\) take values in \(\{y_{12}|y_{12} = 1 + 0.2 \times i, \text{for } i = 1, 2, \ldots, 10\}\); give a partition \(\Pi_{12}\) on interval \([2, 3]\) with step 0.1 and let random variable \(\xi_{13}\) take values in \(\{y_{13}|y_{13} = 2 + 0.1 \times j, \text{for } j = 1, 2, \ldots, 10\}\). For any \(y_{12} \in \Pi_{12}, y_{13} \in \Pi_{13}\), given \(x \in (0.1, 0.2, \ldots, 0.9)\), we calculate \(F^{-1}(x; y_{12}, y_{13})\) by formula (5). Table 1 lists the value of \(F^{-1}(x; y_{12}, y_{13})\).
We use linear interpolation to obtain a continuous function of\ interpolation. To be more precise, fixing \( a \);

\[
F(x; y_{12}, y_{13}) = \int_{0}^{\infty} \int_{0}^{\infty} F(x; y_{12}, y_{13}) dG_{12}(y_{12}) dG_{13}(y_{13})
\]

\[
= \frac{1}{2} \int_{0}^{1} \int_{1}^{3} F(x; y_{12}, y_{13}) dy_{12} dy_{13}
\]

\[
= \frac{1}{2} \sum_{i=1}^{10} \sum_{j=1}^{10} F(x; 1 + 0.2 \ast i, 2 + 0.1 \ast j) + 0.2 \ast 0.1.
\]

The chance distribution function of the minimum length from node 1 to node 4 is shown in Fig. 2.

4. Models of the shortest path in uncertain random networks

In this section, we propose two concepts of shortest path in uncertain random networks and then investigate their properties.

A path from the source node to the destination node is represented by \( p = \{x_{ij}, (i, j) \in U \cup R\} \), where \( x_{ij} = 0 \) indicates that arc \((i, j)\) is not in the path and \( x_{ij} = 1 \) indicates that arc \((i, j)\) is in the path. Then the length of the path \( p = \{x_{ij}, (i, j) \in U \cup R\} \) is

\[
\ell(p) = \sum_{(i,j) \in U \cup R} x_{ij} z_{ij}.
\]

It is clear that \( \ell(p) \) is an uncertain random variable. Since uncertain random variables cannot be directly compared, we propose the following two concepts of shortest path in an uncertain random network:

**Definition 6.** Let \((\mathcal{N}, U, R, \xi)\) be an uncertain random network, and let \( \Phi(z) \) be the chance distribution function of the minimum length from the source node to the destination node. Assume \( P \) is the set of paths from the source node to the destination node, and for each \( p \in P \), the chance distribution function of its length is \( \Psi_p(z) \). The **type-I deviation** between the length of path \( p \) and the minimum length is defined as

\[
\text{Dev}_1(p) = \int_{0}^{\infty} \left\{ \Phi(z) - \Psi_p(z) \right\} dz.
\]

For path \( p' \in P \), if

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<td>2.1</td>
<td>0.3</td>
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</tr>
<tr>
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<td>2.1</td>
<td>0.3</td>
<td>5.7</td>
</tr>
<tr>
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<td>2.1</td>
<td>0.9</td>
<td>5.9</td>
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<tr>
<td>900</td>
<td>10</td>
<td>10</td>
<td>3.0</td>
<td>3.0</td>
<td>0.9</td>
<td>6.9</td>
</tr>
</tbody>
</table>

Fixing \( y_{12} \in \Pi_{12} \) and \( y_{13} \in \Pi_{13} \), we can obtain the uncertain distribution function \( F(x; y_{12}, y_{13}) \) of \( F^{-1}(x; y_{12}, y_{13}) \) via linear interpolation. To be more precise, fixing \( i \) and \( j \), let \( x \) take values from \( \{0.1, 0.2, \ldots, 0.9\} \), then we obtain \( F^{-1}(x; i, j) = x_k, k = 1, 2, \ldots, 9 \). We use linear interpolation to obtain a continuous function of the uncertain distribution function \( F(x; 1 + 0.2 \ast i, 2 + 0.1 \ast j) \) as follows:

\[
F(x; 1 + 0.2 \ast i, 2 + 0.1 \ast j) = \begin{cases} 0, & \text{if } x \leq x_i \\ x_k + (x_{k+1} - x_k) \frac{(x-x_k)}{(x_{k+1}-x_k)}, & \text{if } x_k \leq x \leq x_{k+1}, 1 \leq k \leq 9 \\ 1, & \text{if } x \geq x_9 \end{cases}
\]

where \( x_k = 0.1 \ast k \).

At last, we obtain the chance distribution function of the minimum length, i.e.,

\[
\Phi(x) = \int_{0}^{\infty} \int_{0}^{\infty} F(x; y_{12}, y_{13}) dG_{12}(y_{12}) dG_{13}(y_{13})
\]

\[
= \frac{1}{2} \int_{0}^{1} \int_{1}^{3} F(x; y_{12}, y_{13}) dy_{12} dy_{13}
\]

\[
= \frac{1}{2} \sum_{i=1}^{10} \sum_{j=1}^{10} F(x; 1 + 0.2 \ast i, 2 + 0.1 \ast j) + 0.2 \ast 0.1.
\]

Fig. 2. The chance distribution function of the minimum length.
\[
\int_0^{+\infty} \{\Phi(z) - \Psi_p(z)\} dz = \min_{p \in P} \int_0^{+\infty} \{\Phi(z) - \Psi_p(z)\} dz
\]

then path \( p' \) is called \textbf{type-I shortest path} in the uncertain random network.

\textbf{Definition 7.} Let \((\mathcal{N}, \mathcal{U}, \mathcal{R}, \xi)\) be an uncertain random network, and let \( \Phi(z) \) be the chance distribution function of the minimum length from the source node to the destination node. Assume \( P \) is the set of paths from the source node to the destination node, and for each \( p \in P \), the chance distribution function of its length is \( \Psi_p(z) \). The \textbf{type-II deviation} between the length of path \( p \) and the minimum length is defined as

\[
Dev_{\psi_2}(p) = \max_{0 \leq z < +\infty} \{\Phi(z) - \Psi_p(z)\}.
\]

For path \( p' \in P \), if

\[
\max_{0 \leq z < +\infty} \{\Phi(z) - \Psi_p(z)\} = \min_{p \in P} \max_{0 \leq z < +\infty} \{\Phi(z) - \Psi_p(z)\},
\]

then the path \( p' \) is called \textbf{type-II shortest path} in the uncertain random network.

\textbf{Remark 1.} It should be pointed out that type-I deviation measures the accumulative deviation between chance distribution function \( \Phi \) and chance distribution function \( \Psi_p \), while type-II deviation measures the maximum deviation between \( \Phi \) and \( \Psi_p \). Generally, if path \( p \) has a larger type-I deviation than path \( p' \), then \( p \) has a larger type-II deviation, and vice versa.

We still take the uncertain random network \((\mathcal{N}, \mathcal{U}, \mathcal{R}, \xi)\) presented in Example 1 as an example. There are two paths from node 1 to node 4, namely, Path1 \((1 \rightarrow 2 \rightarrow 4)\) and Path2 \((1 \rightarrow 3 \rightarrow 4)\). By Algorithm 1, we can obtain the chance distribution function of minimum length from node 1 to node 4, i.e., \( \Phi \). Similarly, we can obtain the chance distribution function of Path1 and Path2, i.e., \( \Psi_1 \) and \( \Psi_2 \), respectively. Fig. 3 shows the chance distribution function of minimum length, the length of Path1 and the length of Path2. In Fig. 3, the symbol “M.L.” represents “the minimum length”.

According to Definition 6, we find that the type-I shortest path is Path1 with 1.27, i.e.,

\[
\int_0^{+\infty} \{\Phi(z) - \Psi_1(z)\} dz = 1.27,
\]

\[
\int_0^{+\infty} \{\Phi(z) - \Psi_2(z)\} dz = 6.22.
\]

Obviously, the first integral is smaller than the second one.

According to Definition 7, we find that the type-II shortest path is Path1 with 0.11, i.e.,

\[
\max_{0 \leq z < +\infty} \{\Phi(z) - \Psi_1(z)\} = 0.11,
\]

\[
\max_{0 \leq z < +\infty} \{\Phi(z) - \Psi_2(z)\} = 0.38.
\]

Obviously, the first “max” value is smaller than the second one.

Typically, \( p = \{x_{ij}, (i, j) \in \mathcal{U} \cup \mathcal{R}\} \) is a path from source node 1 to destination node \( n \) if and only if

\[
\begin{cases}
\sum_{j: (i, j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} - \sum_{j: (i, j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} = \begin{cases} 1, & i = 1 \\ 0, & 2 \leq i \leq n - 1 \\ -1, & i = n \end{cases} \\
x_{ij} = \{0, 1\}, & (i, j) \in \mathcal{U} \cup \mathcal{R} \end{cases} \tag{6}
\]

According to Definition 6, the type-I shortest path \( p' = \{x_{ij}, (i, j) \in \mathcal{U} \cup \mathcal{R}\} \) from node 1 to node \( n \) is the optimal solution to the following model:

\[
\begin{cases}
\min_{\{x_{ij}, (i, j) \in \mathcal{U} \cup \mathcal{R}\}} \int_0^{+\infty} (\Phi(z) - \Psi_p(z)) dz \\
\text{s.t.} \\
\sum_{j: (i, j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} - \sum_{j: (i, j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} = \begin{cases} 1, & i = 1 \\ 0, & 2 \leq i \leq n - 1 \\ -1, & i = n \end{cases} \\
x_{ij} = \{0, 1\}, & (i, j) \in \mathcal{U} \cup \mathcal{R} \end{cases} \tag{7}
\]

where \( \Phi(z) \) is the chance distribution function of the minimum length from node 1 to node \( n \) and \( \Psi_p(z) = \text{Ch}\{\sum_{j: (i, j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} \leq z\} \) is chance distribution function of the length of path \( p = \{x_{ij}, (i, j) \in \mathcal{U} \cup \mathcal{R}\} \) from node 1 to node \( n \).

Considering the definition of chance distribution function, the following theorem reformulates model (7).
Theorem 6. Let \( \eta_j \) be independent random variables with probability distribution functions \( G_{\eta_j} \), \((i,j) \in \mathcal{R} \), respectively, and let \( \tau_j \) be independent uncertain variables with uncertainty distribution functions \( Y_j \), \((i,j) \in \mathcal{U} \), respectively. The model (7) can be reformulated as the following model:

\[
\begin{align*}
\min_{(i,j) \in \mathcal{U}} \{ \Phi(z) - \int \int_{z-y} G(y) \} dz \\
\text{s.t.} \\
\sum_{(i,j) \in \mathcal{R}} x_{ij} - \sum_{(i,j) \in \mathcal{R}} x_{ij} = \begin{cases} 
1, & i = 1 \\
0, & 2 \leq i \leq n - 1 \\
-1, & i = n \end{cases} \\
x_{ij} = \{0, 1\}, \quad \forall (i,j) \in \mathcal{U} \cup \mathcal{R} 
\end{align*}
\]

where the distribution functions \( \Phi(z) \), \( G(y) \) and \( Y(r) \) can be obtained by

\[
\Phi(z) = \int_{a}^{b} \cdots \int_{a}^{b} F(z; y_{ij}, (i,j) \in \mathcal{R}) \prod_{(i,j) \in \mathcal{R}} dG_{y_{ij}}(y_{ij}), \\
G(y) = \int_{s}^{c} \sum_{(i,j) \in \mathcal{R}} x_{ij} \prod_{(i,j) \in \mathcal{R}} dG_{x_{ij}}(x_{ij}), \\
Y(r) = \sup_{x_{ij}} \min_{(i,j) \in \mathcal{U}} (x_{ij} r_{ij}).
\]

Proof. According to Theorem 4, the chance distribution function of the minimum length from node 1 to node 1 can be written as

\[
\Phi(z) = \int_{a}^{b} \cdots \int_{a}^{b} F(z; y_{ij}, (i,j) \in \mathcal{R}) \prod_{(i,j) \in \mathcal{R}} dG_{y_{ij}}(y_{ij}),
\]

where \( F(x; y_{ij}, (i,j) \in \mathcal{R}) \) is determined by its inverse uncertainty distribution function

\[
F^{-1}(x; y_{ij}, (i,j) \in \mathcal{R}) = f(c_{ij}, (i,j) \in \mathcal{U} \cup \mathcal{R}),
\]

and \( f \) is calculated by the Dijkstra algorithm for each given \( x \).

Since \( \eta_j \), \((i,j) \in \mathcal{R} \) are independent random variables with probability distribution functions \( G_{\eta_j} \), \((i,j) \in \mathcal{R} \) and \( x_{ij} = \{0, 1\}, \forall (i,j) \in \mathcal{R} \), the probability distribution function of the random variable \( \sum_{(i,j) \in \mathcal{R}} x_{ij} \eta_j \) is

\[
G(y) = \Pr(\eta \leq y) = \int_{s}^{c} \sum_{(i,j) \in \mathcal{R}} x_{ij} \prod_{(i,j) \in \mathcal{R}} dG_{x_{ij}}(x_{ij}).
\]

And since \( \tau_j \) are independent uncertain variables with uncertainty distribution functions \( Y_j \) and \( x_{ij} = \{0, 1\}, \forall (i,j) \in \mathcal{U} \), we have the uncertainty distribution function of the uncertain variable \( \tau = \sum_{(i,j) \in \mathcal{R}} x_{ij} \tau_j \) by Theorem 2. That is

\[
Y(r) = \mathcal{M}(\tau \leq r) = \sup_{x_{ij}} \min_{(i,j) \in \mathcal{U}} (x_{ij} r_{ij}).
\]

According to Definition 1 and Theorem 3, we have

\[
\Psi(z) = \mathcal{C} \left\{ \sum_{(i,j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} \eta_j \leq z \right\} = \mathcal{C} \left\{ \sum_{(i,j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} \tau_j \leq z \right\}
\]

where \( \mathcal{C} \left\{ \sum_{(i,j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} \tau_j \leq z \right\} \) is the chance distribution function of the minimum length from node 1 to node 1.

Further, we can obtain

\[
\Psi(z) = \int_{\mathcal{U} \cup \mathcal{R}} Y \left\{ \sum_{(i,j) \in \mathcal{R}} x_{ij} Y_j \right\} \prod_{(i,j) \in \mathcal{R}} dG_{x_{ij}}(x_{ij})
\]

where \( \sum_{(i,j) \in \mathcal{R}} x_{ij} Y_j \) denotes numbers of random edges.

The proof is completed. \( \square \)

According to Definition 7, the type-II shortest path \( p_n^*(x_{ij}) = \{x_{ij}, (i,j) \in \mathcal{U} \cup \mathcal{R} \} \) from node 1 to node \( n \) is the optimal solution to the following model:

\[
\begin{align*}
\min_{(i,j) \in \mathcal{U} \cup \mathcal{R}} \max_{0 \leq x \leq \infty} \left\{ \Phi(z) - \int \int_{z-y} G(y) \right\} \\
\text{s.t.} \\
\sum_{(i,j) \in \mathcal{R}} x_{ij} - \sum_{(i,j) \in \mathcal{R}} x_{ij} = \begin{cases} 
1, & i = 1 \\
0, & 2 \leq i \leq n - 1 \\
-1, & i = n \end{cases} \\
x_{ij} = \{0, 1\}, \quad \forall (i,j) \in \mathcal{U} \cup \mathcal{R} 
\end{align*}
\]

where \( \Phi(z) \) is the chance distribution function of the minimum length and \( \Psi(z) = \mathcal{C} \left\{ \sum_{(i,j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} \tau_j \leq z \right\} \) is chance distribution function of the length of path \( \{x_{ij}, (i,j) \in \mathcal{U} \cup \mathcal{R} \} \) from node 1 to node \( n \).

Similarly, the following theorem reformulates model (9).

Theorem 7. The model (9) can be reformulated as the following model:

\[
\begin{align*}
\min_{(i,j) \in \mathcal{U} \cup \mathcal{R}} \max_{0 \leq x \leq \infty} \left\{ \Phi(z) - \int \int_{z-y} G(y) \right\} \\
\text{s.t.} \\
\sum_{(i,j) \in \mathcal{R}} x_{ij} - \sum_{(i,j) \in \mathcal{R}} x_{ij} = \begin{cases} 
1, & i = 1 \\
0, & 2 \leq i \leq n - 1 \\
-1, & i = n \end{cases} \\
x_{ij} = \{0, 1\}, \quad \forall (i,j) \in \mathcal{U} \cup \mathcal{R} 
\end{align*}
\]

where the distribution functions \( \Phi(z) \), \( G(y) \) and \( Y(r) \) can be obtained by

\[
\Phi(z) = \int_{a}^{b} \cdots \int_{a}^{b} F(z; y_{ij}, (i,j) \in \mathcal{R}) \prod_{(i,j) \in \mathcal{R}} dG_{y_{ij}}(y_{ij}), \\
G(y) = \int_{s}^{c} \sum_{(i,j) \in \mathcal{R}} x_{ij} \prod_{(i,j) \in \mathcal{R}} dG_{x_{ij}}(x_{ij}), \\
Y(r) = \sup_{x_{ij}} \min_{(i,j) \in \mathcal{U}} (x_{ij} r_{ij}).
\]
In order to find the type-I and type-II shortest path in uncertain random network \((N, \mathcal{U}, R, \xi)\), we propose the following algorithm.

**Algorithm 2.**

**Step 1.** By Algorithm 1, calculate and save the value of chance distribution function of the minimum length from node 1 to node \(n\).

**Step 2.** Using the breadth first search algorithm (BFS), obtain all paths in the uncertain random network.

**Step 3.** By Theorems 6 and 7, calculate type-I and type-II deviations between the chance distribution function of minimum length and that of each path obtained in Step 2 respectively.

**Step 4.** Compare the deviations, and choose the minimum value, whose corresponding path is the shortest path.

**5. Numerical example**

In this section, we give an example to illustrate the algorithms presented above. The uncertain random network \((N, \mathcal{U}, R, W)\) is shown in Fig. 4, with 10 nodes and 15 arcs.

Denote the chance distribution function of the minimum length from node 1 to node 10 as \(\Phi(z)\). The arc information is listed in Table 2, in which \(z_{ij}\) are random variables, \(\eta_{ij}\) are uncertain variables. In Table 2, \(U(a, b)\) represents uniformly probability distribution function, \(L(a, b)\) represents linear uncertainty distribution function and \(Z(a, b, c)\) represents zigzag uncertainty distribution function. If the length of arc \((i, j)\) is a constant, we simply regard it as a special uncertain variable. For example, the length of arc \((5, 8)\) is 15, which is regarded as an uncertain variable.

Using the breadth first search algorithm, we can find all paths from node 1 to node 10, which are listed in the second column of Table 3. The chance distribution functions of these paths can be obtained, which are shown in Fig. 5.

According to Theorems 6 and 7, we calculate the type-I and type-II deviations of these paths, which are presented in the third column and the fourth column of Table 3 respectively. According to Definition 6, the type-I shortest path is Path4, i.e., \(1 \rightarrow 3 \rightarrow 6 \rightarrow 10\), the type-I deviation of which is 0.44. According to Definition 7, the type-II shortest path is Path4, i.e., \(1 \rightarrow 3 \rightarrow 6 \rightarrow 10\), the type-II deviation of which is 0.15.

If we sort the paths in column 1 of Table 3 in the ascending order of type-I deviation and type-II deviation respectively, we can obtain the same sequence of paths, which is consistent with Remark 1.

![Fig. 4. An uncertain random network \((N, \mathcal{U}, R, \xi)\).](image)

![Fig. 5. The chance distribution function of all paths.](image)

**Table 2**

<table>
<thead>
<tr>
<th>Path</th>
<th>Path line</th>
<th>Type-I deviation</th>
<th>Type-II deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path1</td>
<td>1 → 2 → 5 → 8 → 10</td>
<td>4.48</td>
<td>0.92</td>
</tr>
<tr>
<td>Path2</td>
<td>1 → 3 → 5 → 8 → 10</td>
<td>1.83</td>
<td>0.57</td>
</tr>
<tr>
<td>Path3</td>
<td>1 → 3 → 6 → 8 → 10</td>
<td>3.54</td>
<td>0.69</td>
</tr>
<tr>
<td>Path4</td>
<td>1 → 3 → 6 → 10</td>
<td>0.44</td>
<td>0.15</td>
</tr>
<tr>
<td>Path5</td>
<td>1 → 3 → 6 → 9 → 10</td>
<td>10.18</td>
<td>0.99</td>
</tr>
<tr>
<td>Path6</td>
<td>1 → 3 → 7 → 9 → 10</td>
<td>4.39</td>
<td>0.90</td>
</tr>
<tr>
<td>Path7</td>
<td>1 → 4 → 7 → 9 → 10</td>
<td>1.03</td>
<td>0.33</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>Paths</th>
<th>Path line</th>
<th>Type-I deviation</th>
<th>Type-II deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Path1</td>
<td>1 → 2 → 5 → 8 → 10</td>
<td>4.48</td>
<td>0.92</td>
</tr>
<tr>
<td>Path2</td>
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<td>1.83</td>
<td>0.57</td>
</tr>
<tr>
<td>Path3</td>
<td>1 → 3 → 6 → 8 → 10</td>
<td>3.54</td>
<td>0.69</td>
</tr>
<tr>
<td>Path4</td>
<td>1 → 3 → 6 → 10</td>
<td>0.44</td>
<td>0.15</td>
</tr>
<tr>
<td>Path5</td>
<td>1 → 3 → 6 → 9 → 10</td>
<td>10.18</td>
<td>0.99</td>
</tr>
<tr>
<td>Path6</td>
<td>1 → 3 → 7 → 9 → 10</td>
<td>4.39</td>
<td>0.90</td>
</tr>
<tr>
<td>Path7</td>
<td>1 → 4 → 7 → 9 → 10</td>
<td>1.03</td>
<td>0.33</td>
</tr>
</tbody>
</table>
6. Conclusions

In an uncertain random network, some arc lengths are uncertain variables while some are random variables. Under the framework of chance theory, this paper designed an algorithm to calculate the chance distribution function of minimum length from the source node to the destination node, which was only theoretically formulated in existing literature. Since the traditional concept of the shortest path is no longer suitable in uncertain random networks, this paper proposed two types of shortest path in uncertain random networks, namely type-I and type-II shortest paths, the definition of which depends on the deviation between chance distribution functions. Two optimization models were constructed for type-I and type-II shortest path in an uncertain random network, respectively. In order to solve these models and find the optimal paths, an algorithm was designed, the effectiveness of which was illustrated by numerical examples. Besides, the consistency of type-I and type-II shortest paths is also verified numerically.

The main drawback of this paper is that the first step of the algorithm for finding type-I and type-II shortest paths is actually enumerating all the paths from the source node to the destination node, which is inefficient. For this reason, the methods proposed in this paper can only handle networks with small size. In the future research, designing an efficient algorithm to find the paths will be studied. Another drawback is that we don’t distinguish risk-averse decision-makers and risk-prone decision-makers. In practice, different decision-makers have different perspectives on uncertainty, which may be considered in the future research. Besides, we also assume that all the uncertain arc lengths and the random arc lengths are independent in this paper, which may be relaxed in future models.

Acknowledgements

The first author was supported by the National Natural Science Foundation of China (Nos. 61462086 and 61563050), and by Xinjiang University No. BS150206. The second author was supported by the National Natural Science Foundation of China (Nos. 71401008, 71401007 and 71571018), State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University (No. RCS2015ZZ003), and China Scholarship Council.

References