Some results of moments of uncertain set

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Abstract. Uncertain set theory is a generalization of uncertainty theory that has become a new branch of mathematics for modeling human belief degrees. Uncertain set is a fundamental concept to describe unsharp concepts in uncertain set theory. The moments are important characteristics of an uncertain set. This paper studies the moments and central moments of uncertain set and gives some formulas to calculate the moments and central moments by membership function. In addition, some examples are provided to calculate the moments and central moments of uncertain set.

Keywords: Uncertainty theory, uncertain set, membership function, moments

1. Introduction

Since Kolmogorov [6] established probability theory in 1933, probability theory has become a branch of mathematics for modeling random phenomena. Before applying probability theory in practice, we should obtain probability distribution is close enough to the real frequency via statistics. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable. In the real world, we sometimes have no observed data because of the technological or economical difficulties. In this case, we have to invite domain experts to estimate their belief degree that each event will happen, but, human beings usually estimate a much wider range of values than the object actually takes (Liu [7]). This conservatism of human beings makes the belief degrees deviate far from the frequency. If we still take human belief degrees as probability distribution, we may cause a counterintuitive result that was given by Liu [15].

In order to model human belief degrees, uncertainty theory was established by Liu [7] in 2007 and refined by Liu [12] in 2010. Uncertainty theory has become a new branch of mathematics for modeling indeterminate phenomena based on normality, duality, subadditivity and product axioms. As a fundamental concept in uncertainty theory, Liu [7] proposed the concept of uncertain variable and presented the uncertainty distribution of uncertain variable to describe uncertain variable in practice. In order to rank uncertain variables, the concept of expected value operator was proposed by Liu [7], which is the average value of uncertain variable in the sense of uncertain measure. Liu [7] also presented the concept of variance that provides a degree of the spread of the distribution around its expected value. Recently, Yao [30] proposed a formula to calculate the variance using inverse uncertainty distribution. Based on the expected operator, Liu [7] presented the concepts of moments of uncertain variable and gave a formula to calculate the moments using inverse uncertainty distribution. Further, Sheng and Kar [22] proposed a formula to calculate the moments using inverse uncertainty distribution. In the last few years, many researchers had important contributions in uncertainty theory area, for example, uncertain programming (Liu [10], Liu [12]), uncertain risk analysis (Liu [13], Peng [20]), uncertain differential equation (Liu [8], Chen and Liu [1], Yao and Chen [27]), uncertain finance (Liu [9], Chen and...
Gao [2], Liu et al. [19]), uncertain optimal control (Zhu [31]), uncertain game (Gao [3], Yang and Gao [24–26]), etc. For more detailed exposition of uncertainty theory with applications, the readers may consult Liu’s recent book [18].

In 2010, Liu [11] first proposed uncertain set in order to model unsharp concepts that are essentially sets but their boundaries are not sharply described. In order to measure uncertain set, the membership function and inverse membership function of uncertain set were presented by Liu [16]. Besides, Liu [17] gave the definition of independence for uncertain set, and Liu [16] provided a set operational law of uncertain sets via membership function, and an arithmetic operational law via inverse membership function.

As an important feature of uncertain set, the expected value of uncertain set was proposed by Liu [11]. After then, Liu [12] gave a formula to calculate the expected value via membership function, and Liu [16] gave another formula via inverse membership function. In addition, the concept of variance and distance between uncertain sets were presented by Liu [14]. In order to measure the uncertainty of uncertain set, Liu [14] first defined entropy of uncertain set. As extensions of entropy, Wang and Ha [23] proposed a quadratic entropy, Yao and Ke [28] suggested a cross entropy for comparing a membership function against a reference membership function and Yao [29] presented sine entropy of uncertainty to apply in portfolio selection. A questionnaire survey for collecting expert’s experimental date was presented by Liu [14].


The rest of this paper is organized as follows. In Section 2, we briefly review some basic results of uncertainty theory and uncertain set. Section 3 presents moments and central moments of uncertain set. Meanwhile, we give some formulas to calculate moment and central moment of uncertain set by membership function. In addition, some practical examples are provided to calculate the moments and central moments of uncertain set.

2. Preliminaries

In this section, we first review some basic concepts of uncertainty theory and then introduces some results of uncertain set. Uncertain measure $\mathcal{M}$ is a real-valued set-function on a $\sigma$-algebra $\mathcal{L}$ over a nonempty set $\Gamma$ satisfying normality, duality, subadditivity and product axioms. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2.1. (Liu [7]) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set $\Gamma$.

Axiom 2. (Duality Axiom) $\mathcal{M}(A^c) + \mathcal{M}(A) = 1$ for any event $A$.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $A_1, A_2, \cdots$, we have

$$\mathcal{M}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mathcal{M}(A_k).$$

Besides, in order to provide the operational law, Liu [9] defined the product uncertain measure on the product $\sigma$-algebra $\mathcal{L}$ as follows.

Axiom 4. (Product Axioms) Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left(\prod_{k=1}^{\infty} A_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}(A_k),$$

where $A_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2.2. (Liu [7]) An uncertain variable is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, such that, for any Borel set $B$ of real numbers, the set

$$\{x \in B \} = \{y \in \Gamma | (x, y) \in B\}$$

is an event.
In 2010, Liu [11] first proposed an uncertain set to model unsharp concepts that are essentially sets but whose boundaries are not sharply described.

**Definition 2.3.** (Liu [11]) An uncertain set is a function ξ from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to a collection of sets such that
\[
\{B \subset \xi\} = \{y \in \Gamma \mid B \subset \xi(y)\},
\]
\[
\{x \subset B\} = \{y \in \Gamma \mid \xi(y) \subset B\}
\]
are events for any Borel set \(B\).

**Remark 2.1.** Roughly speaking, an uncertain set is a set-valued function on an uncertainty space, while an uncertain variable is a real-valued function on an uncertainty space. That is, the uncertain variable refers to one value, while the uncertain set to a collection of values.

In order to describe uncertain set in practice, Liu [16] defined membership function in 2012.

**Definition 2.4.** (Liu [16]) An uncertain set \(\xi\) is said to have a membership function \(\mu\) if for any Borel set \(B\) of real numbers, we have
\[
\mathcal{M}(B \subset \xi) = \inf_{x \in B} \mu(x),
\]
\[
\mathcal{M}(\xi \subset B) = 1 - \sup_{x \in B} \mu(x).
\]
The above equations will be called measure inversion formulas.

For example, let the uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) be \([0, 1]\) with Borel algebra and Lebesgue measure. For each \(y \in [0, 1]\), define \(\xi(y) = [y - 1, 1 - y]\). Then \(\xi\) is an uncertain set with a membership function
\[
\mu(x) = \begin{cases} 
1 - |x|, & \text{if } x \in [-1, 1] \\
0, & \text{otherwise}
\end{cases}
\]
An uncertain set \(\xi\) is called rectangular if it has a membership function
\[
\mu(x) = \begin{cases} 
1, & \text{if } a \leq x \leq b \\
0, & \text{otherwise}
\end{cases}
\]
denoted by \((a, b)\) where \(a\) and \(b\) are real numbers with \(a < b\). An uncertain set \(\xi\) is called triangular if it has a membership function
\[
\mu(x) = \begin{cases} 
x - a & \text{if } a \leq x \leq b \\
b - a & \text{otherwise}
\end{cases}
\]
denoted by \((a, b, c)\) where \(a, b, c\) are real numbers with \(a < b < c\). An uncertain set \(\xi\) is called trapezoidal if it has a membership function
\[
\mu(x) = \begin{cases} 
x - a & \text{if } a \leq x \leq b \\
b - a & \text{otherwise}
\end{cases}
\]
denoted by \((a, b, c, d)\) where \(a, b, c, d\) are real numbers with \(a < b < c < d\).

A membership function \(\mu\) is said to be regular if there exists a point \(x_0\) such that \(\mu(x_0) = 1\) and \(\mu(x)\) is unimodal about the mode \(x_0\).

**Definition 2.5.** (Liu [16]) Let \(\xi\) be an uncertain set with membership function \(\mu\). Then the set-valued function
\[
\mu^{-1}(a) = \{x \in \mathbb{R} \mid \mu(x) \geq a\}
\]
is called the inverse membership function of \(\xi\). Sometimes, the set \(\mu^{-1}(a)\) is called the \(a\)-cut of \(\mu\).

**Definition 2.6.** (Liu [16]) The uncertain sets \(\xi_1, \xi_2, \ldots, \xi_n\) are said to be independent if for any Borel sets \(B_1, B_2, \ldots, B_n\), we have
\[
\mathcal{M}\left(\bigcap_{i=1}^{n} \{\xi_i \subset B_i\}\right) = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \subset B_i\}
\]
and
\[
\mathcal{M}\left(\bigcup_{i=1}^{n} \{\xi_i \subset B_i\}\right) = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \subset B_i\}
\]
where \(\xi_i^\gamma\) are arbitrarily chosen from \([\xi_i, \xi_i]\), \(i = 1, 2, \ldots, n\), respectively.

The union and intersection of uncertain sets \(\xi\) and \(\eta\) are defined as \((\xi \cup \eta)(y) = \xi(y) \cup \eta(y)\) and \((\xi \cap \eta)(y) = \xi(y) \cap \eta(y)\) for every \(y \in \Gamma\), respectively. And complement of uncertain sets \(\xi\) is defined as \(\xi^c(y) = \xi(y)^c\) for every \(y \in \Gamma\). They are also uncertain sets. For the set operations of uncertain sets, Liu [16] proved the following theorem using membership function.

**Theorem 2.1.** (Liu [16]) Let \(\xi\) and \(\eta\) be independent uncertain sets with membership functions \(\mu\) and \(\nu\), respectively. Then the union \(\xi \cup \eta\) has a membership function
\[
\lambda_1(x) = \mu(x) \lor \nu(x),
\]
the intersection \(\xi \cap \eta\) has a membership function
\[
\lambda_2(x) = \mu(x) \land \nu(x),
\]
the complement $\xi^c$ has a membership function

$$\lambda^c(x) = 1 - \mu(x).$$

Moreover, for the arithmetic operations of uncertain sets, Liu [16] proved the following theorem using inverse membership function.

**Theorem 2.2.** (Liu [16]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_n^{-1}$, respectively. If $f$ is a measurable function, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain set with inverse membership function,

$$\lambda^{-1}(a) = f(\mu_1^{-1}(a), \mu_2^{-1}(a), \ldots, \mu_n^{-1}(a)).$$

An uncertain set $\xi$ is said to be nonempty if $\xi(\gamma) \neq \emptyset$ for almost all $\gamma \in \Gamma$. Liu [11] proposed the concept of expected for nonempty uncertain set.

**Definition 2.7.** (Liu [11]) Let $\xi$ be a nonempty uncertain set. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{-\infty}^{+\infty} M(\xi \geq x) dx - \int_{-\infty}^{+\infty} M(\xi \leq x) dx$$

provided that at least one of the two integrals is finite.

Note that $E[\xi] \geq x$ represents $\xi$ is imaginarily included in $(x, +\infty)$, and $\xi \leq x$ represents $\xi$ is imaginarily included in $(-\infty, x]$.

**Theorem 2.3.** (Liu [12]) Let $\xi$ be an uncertain set with regular membership function $\mu$. Then

$$E[\xi] = x_0 + \frac{1}{2} \int_{-\infty}^{+\infty} \mu(x) dx - \frac{1}{2} \int_{-\infty}^{+\infty} \mu(x) dx$$

where $x_0$ is a point such that $\mu(x_0) = 1$.

The expected value of rectangular uncertain set $\xi = (a, b)$ is $E[\xi] = (a + b)/2$. The expected value of triangular uncertain set $\xi = (a, b, c)$ is $E[\xi] = (a + 2b + c)/4$. The expected value of trapezoidal uncertain set $\xi = (a, b, c, d)$ is $E[\xi] = (a + b + c + d)/4$.

**Theorem 2.4.** (Liu [16]) Let $\xi$ and $\eta$ be independent uncertain sets with finite expected values. Then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

The variance of uncertain set that was proposed by Liu [14] provides a degree of the spread of the membership function around its expected value.

**Definition 2.8.** (Liu [14]) Let $\xi$ be an uncertain set with finite expected value $\mu$. Then the variance of $\xi$ is defined by

$$V[\xi] = E[(\xi - \mu)^2]$$

**Theorem 2.5.** (Liu [7]) Let $\xi$ be an uncertain set with membership function $\mu$ and expected value $\mu$. Then the variance of $\xi$ is

$$V[\xi] = \frac{1}{2} \int_{-\infty}^{+\infty} \sup_{0 < y < x} \mu(y) + \sup_{0 < y < x} \mu(y) dx.$$

The variance of rectangular uncertain set $(a, b)$ is $V[\xi] = (b - a)^2/8$. The variance of symmetric triangular uncertain set $(a, b, c)$ is $V[\xi] = (c - a)^2/24$.

### 3. Moments

In this section, we define the concept of moments of uncertain set and give some formulas to calculate the moments of uncertain set by membership function.

**Definition 3.1.** Let $\xi$ be an uncertain set and let $k$ be a positive integer. If $E[\xi^k]$ is finite, then $E[\xi^k]$ is called the $k$-th moment of $\xi$.

This definition says that the $k$-th moment of uncertain set $\xi$ is just the expected value of $\xi^k$. From Definition 2.7, we have

$$E[\xi^k] = \int_{-\infty}^{+\infty} M(\xi^k \geq x) dx - \int_{-\infty}^{+\infty} M(\xi^k \leq x) dx$$

or simply

$$E[\xi^k] = \int_{-\infty}^{+\infty} M[\xi^k \geq x] dx - \int_{-\infty}^{+\infty} M[\xi^k \leq x] dx$$

In order to calculate $k$-th moment of $\xi$ by (1), we must determine the value of $M[\xi^k \geq x]$ and $M[\xi^k \leq x]$. Intuitively, for $M[\xi^k \geq x]$, it is too conservative if we take value $M[\xi^k \geq x]$ and is too adventurous if we take the value $1 - M[\xi^k < x]$. Thus we assign $M[\xi^k \geq x]$ as the middle value between $M[\xi^k \geq x]$ and $1 - M[\xi^k < x]$. That is

$$M[\xi^k \geq x] = \frac{1}{2} \left( M[\xi^k \geq x] + 1 - M[\xi^k < x] \right),$$

$$M[\xi^k \leq x] = \frac{1}{2} \left( M[\xi^k \leq x] + 1 - M[\xi^k < x] \right).$$

**Theorem 3.1.** Let $\xi$ be a nonempty uncertain set with membership function $\mu$, and let $k$ be a positive integer.
Then for any number $x$, we have

$$M[\xi \geq x] = \frac{1}{2} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right). \quad (2)$$

$$M[\xi \leq x] = \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right). \quad (3)$$

**Proof:** Firstly, assume $k$ is an odd number. Since $\xi$ is an uncertain set with membership function $\mu$, it follows from the measure inversion formula that for any real number $x$, we have

$$M[\xi \geq x] = M[\xi \geq \sqrt{k}] = M[\xi \subset (-\infty, \sqrt{k}]) = 1 - \sup_{y < \sqrt{k}} \mu(y) = \frac{1}{2} \left( \sup_{y \geq \sqrt{k}} \mu(y) + 1 - \sup_{y < \sqrt{k}} \mu(y) \right).$$

Thus,

$$M[\xi \geq x] = \frac{1}{2} \left( M[\xi \geq x] + 1 - M[\xi < x] \right) = \frac{1}{2} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right).$$

The equation (2) is true. We may also prove (3) similarly.

Secondly, assume $k$ is an even number. If $x \geq 0$, we have

$$M[\xi \geq x] = M[\xi \subset (-\infty, -\sqrt{k}) \cup \{\sqrt{k}, +\infty\}] = 1 - \sup_{y < \sqrt{k}} \mu(y) = \frac{1}{2} \left( \sup_{y \geq \sqrt{k}} \mu(y) + 1 - \sup_{y < \sqrt{k}} \mu(y) \right).$$

If $x < 0$, we have

$$M[\xi \geq x] = 1 - \sup_{y < x} \mu(y), \quad M[\xi < x] = 0 - \sup_{y < x} \mu(y).$$

Thus, for any real number $x$, we have

$$M[\xi \geq x] = \frac{1}{2} \left( M[\xi \geq x] + 1 - M[\xi < x] \right) = \frac{1}{2} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right).$$

The equation (2) is true. We may also prove (3) similarly. The theorem is thus proved. \( \square \)

**Theorem 3.2.** Let $\xi$ be a nonempty uncertain set with membership function $\mu$, and let $k$ be a positive integer. Then the $k$-th moment of $\xi$ is

$$E[\xi^k] = \frac{1}{2} \int_{-\infty}^{+\infty} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right) \mu(\xi) \, dx.$$

**Proof:** This theorem follows from (1) and Theorem 3.1 immediately. \( \square \)

**Theorem 3.3.** Let $\xi$ be an uncertain set with regular membership function $\mu$, and let $k$ be an odd number. Then the $k$-th moment of $\xi$ is

$$E[\xi^k] = \frac{1}{2} \int_{-\infty}^{+\infty} \left( \sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y) \right) \mu(\xi) \, dx,$$

where $x_0$ is a point such that $\mu(x_0) = 1$.

**Proof:** Since $\mu$ is increasing on $(-\infty, x_0)$ and decreasing on $[x_0, +\infty)$, from (2) and (3), we obtain

$$M[\xi \geq x] = \begin{cases} 1 - \mu(\sqrt{k}/2), & \text{if } x \leq x_0^k, \\ \mu(\sqrt{k}/2), & \text{if } x \geq x_0^k. \end{cases}$$
Theorem 3.4. Let $ξ$ be a positive uncertain set with regular membership function $μ$, and let $k$ be an even number. Then the $k$-th moment of $ξ$ is

$$E[ξ^k] = \int_{0}^{\infty} \frac{1}{2} \left(1 - μ(\sqrt{3}x)\right) dx + \int_{-\infty}^{0} \frac{1}{2} \left(1 - μ(\sqrt{3}x)\right) dx \quad \text{if} \quad a > b \quad \text{or} \quad b > a$$

where $x_0$ is a point such that $μ(x_0) = 1$. If $ξ$ is negative, then the above equation is also true.

Proof: Firstly, when $ξ$ is a positive uncertain set, we have $x_0 > 0$. Since $μ$ is increasing on $[0, x_0]$ and decreasing on $(x_0, +\infty)$, we obtain

$$M[ξ^k ≥ x] = \begin{cases} \frac{1}{2} \mu(\sqrt{3}x/2), & \text{if } x ≤ x_0^k \\ \frac{1}{2} \mu(\sqrt{3}x/2), & \text{if } x ≥ x_0^k \end{cases}$$

for any real number $x$ from (2).

Then

$$E[ξ^k] = \int_{0}^{\infty} M[ξ^k ≥ x] dx$$

$$= \int_{0}^{x_0^k} \frac{1}{2} \left(1 - μ(\sqrt{3}x)\right) dx + \int_{x_0^k}^{\infty} \frac{1}{2} \left(1 - μ(\sqrt{3}x)\right) dx$$

Secondly, when $ξ$ is a negative uncertain set, we have $x_0 < 0$. Since $μ$ is increasing on $(-\infty, x_0)$ and decreasing on $[x_0, 0]$, we obtain

$$M[ξ^k ≥ x] = \begin{cases} \frac{1}{2} \mu(-\sqrt{3}x/2), & \text{if } x ≤ x_0^k \\ \frac{1}{2} \mu(-\sqrt{3}x/2), & \text{if } x ≥ x_0^k \end{cases}$$

for any real number $x$ from (2).

Then

$$E[ξ^k] = \int_{0}^{\infty} M[ξ^k ≥ x] dx$$

$$= \int_{0}^{x_0^k} \frac{1}{2} \left(1 - μ(-\sqrt{3}x)\right) dx + \int_{x_0^k}^{\infty} \frac{1}{2} \left(1 - μ(-\sqrt{3}x)\right) dx$$

The theorem is thus proved. □

Example 3.1. Let $k$ be an even number, and let $ξ$ be an uncertain set with finite expected value $e$. If membership function $μ$ is regular and symmetric, then the $k$-th moment of $ξ$ is

$$E[ξ^k] = e^k - \frac{k}{2} \int_{0}^{\infty} x^{k-1} μ(\text{sgn}(x)) dx$$

Example 3.2. Consider the $k$-th moment of rectangular uncertain set $ξ = (a, b)$. If $k$ is an odd number, then

$$E[ξ^k] = \frac{a^k + b^k}{2}$$
If \( k \) is an even number, then
\[
E[\xi^k] = \begin{cases} 
(a^k + b^k)/2, & \text{if } ab > 0 \\
\max\{a^k, b^k\}/2, & \text{if } a \leq 0, b \geq 0. 
\end{cases}
\]

**Example 3.3.** Consider the \( k \)-th moment of triangular uncertain set \( \xi = (a, b, c), ac \geq 0. \)
Then
\[
E[\xi^k] = \frac{1}{2(k + 1)} \left[ 2b^k + \sum_{i=0}^{k-1} b'_{i+1}(a^{k-i} + c^{k-i}) \right].
\]

**Example 3.4.** Consider the \( k \)-th moment of trapezoidal uncertain set \( \xi = (a, b, c, d), \) \( ac \geq 0. \)
Then
\[
E[\xi^k] = \frac{1}{3k + 1} \left[ b' + c' + \sum_{i=0}^{k-1} b'_i a^{k-i} + c'_i c^{k-i} \right].
\]

4. Central moments

In this section, we define the concept of central moments of uncertain set and give some formulas to calculate the central moments of uncertain set with membership function.

**Definition 4.1.** Let \( \xi \) be an uncertain set with finite expected value \( e \), and let \( k \) be a positive integer. If \( E[(\xi - e)^k] \) is finite, then \( E[(\xi - e)^k] \) is called the \( k \)-th central moment of \( \xi \).

This definition says that the \( k \)-th central moment of uncertain set \( \xi \) is the expected value of \((\xi - e)^k\).

From Definition 2.7, we have
\[
E[(\xi - e)^k] = \int_{-\infty}^{\infty} M((\xi - e)^k \geq x) \, dx - \int_{-\infty}^{0} M((\xi - e)^k \leq x) \, dx.
\]

In order to calculate \( k \)-th central moments of uncertain set \( \xi \) by (5), we must determine the value of \( M((\xi - e)^k \geq x) \) and \( M((\xi - e)^k \leq x) \) from the following formulas,
\[
M((\xi - e)^k \geq x) = \frac{1}{2} \left( M(\xi^k \geq x) + 1 - M(\xi^k < x) \right),
\]
\[
M((\xi - e)^k \leq x) = \frac{1}{2} \left( M(\xi^k \geq x) + 1 - M(\xi^k < x) \right).
\]

**Theorem 4.1.** Let \( \xi \) be a nonempty uncertain set with membership function \( \mu \), and let \( k \) be a positive integer. Then for any number \( x \), we have
\[
M((\xi - e)^k \geq x) = \frac{1}{2} \left( \sup_{(x-e)^k \geq x} \mu(y) + 1 - \sup_{(x-e)^k < x} \mu(y) \right),
\]
\[
M((\xi - e)^k \leq x) = \frac{1}{2} \left( \sup_{(x-e)^k \geq x} \mu(y) + 1 - \sup_{(x-e)^k < x} \mu(y) \right).
\]

**Proof:** Firstly, assume \( k \) is an odd number. Since \( \xi \) is an uncertain set with membership function \( \mu \), it follows from the measure inversion formula that for any real number \( x \), we have
\[
M((\xi - e)^k \geq x) = M(\xi \geq e + \sqrt[k]{x})
\]
\[
= M(\xi \geq e + \sqrt[k]{x})
\]
\[
= M(\xi \cap (e + \sqrt[k]{x}, +\infty))
\]
\[
= 1 - \sup_{(x-e)^k < x} \mu(y),
\]
\[
M((\xi - e)^k \leq x) = M(\xi < e + \sqrt[k]{x})
\]
\[
= M(\xi \subset (-\infty, e + \sqrt[k]{x})]
\]
\[
= M(\xi \subset (-\infty, e + \sqrt[k]{x}))
\]
\[
= 1 - \sup_{(x-e)^k > x} \mu(y),
\]
Thus,
\[
M((\xi - e)^k \geq x) = \frac{1}{2} \left( M((\xi - e)^k \geq x) + 1 - M((\xi - e)^k < x) \right)
\]
\[
= \frac{1}{2} \left( \sup_{(x-e)^k \geq x} \mu(y) + 1 - \sup_{(x-e)^k < x} \mu(y) \right).
\]

The equation (6) is true. We may also prove (7) similarly.
Secondly, assume \( k \) is an even number. If \( x \geq 0 \), we have
\[
M(\xi - e^k \geq x) = M(\xi \subset (-\infty, e - \sqrt{k}) \cup [e + \sqrt{k}, +\infty]) = 1 - \sup_{y \leq x} \mu(y).
\]
\[
M(\xi - e^k < x) = M(\xi \subset (e - \sqrt{k}, e + \sqrt{k})) = 1 - \sup_{y \geq x} \mu(y).
\]
If \( x < 0 \), we have
\[
M(\xi - e^k \geq x) = 1 - \sup_{y \leq x} \mu(y),
\]
\[
M(\xi - e^k < x) = 0 - \sup_{y \geq x} \mu(y).
\]
Thus, for any real number \( x \), we have
\[
M(\xi - e^k \geq x) = 1 - \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y \geq x} \mu(y) \right).
\]
The equation (6) is true. We may also prove (7) similarly. The theorem is thus proved. \qed

**Theorem 4.2.** Let \( \xi \) be a nonempty uncertain set with membership function \( \mu \) and finite expected value \( e \), and let \( k \) be a positive integer. Then the \( k \)-th central moment of \( \xi \) is
\[
E(\xi - e^k) = \frac{1}{2} \int_{-\infty}^{+\infty} \left( \sup_{y \leq e^k} \mu(y) + 1 - \sup_{y \geq e^k} \mu(y) \right) dx
\]
\[
- \frac{1}{2} \int_{-\infty}^{0} \left( \sup_{y \leq e^k} \mu(y) + 1 - \sup_{y \geq e^k} \mu(y) \right) dx.
\]

**Proof:** This theorem follows from (5) and Theorem 4 immediately. \qed

**Theorem 4.3.** Let \( \xi \) be an uncertain set with regular membership function \( \mu \) and finite expected value \( e \), and let \( k \) be an odd number. Then the \( k \)-th central moment of \( \xi \) is
\[
E(\xi - e^k) = (x_0 - e^k)^2 + \frac{k}{2} \int_{x_0}^{+\infty} (x - e^k - 1)^2 \mu(x) dx
\]
\[
- \frac{k}{2} \int_{-\infty}^{x_0} (x - e^k)^2 \mu(x) dx
\]
where \( x_0 \) is a point such that \( \mu(x_0) = 1 \).

**Proof:** Since \( \mu \) is increasing on \((-\infty, x_0]\) and decreasing on \([x_0, +\infty)\), from (6) and (7), we obtain
\[
M(\xi - e^k \geq x) = \begin{cases}
1 - \mu(x + \sqrt{k}/2), & \text{if } x \leq (x_0 - e^k) \\
\mu(x + \sqrt{k}/2), & \text{if } x \geq (x_0 - e^k)
\end{cases}
\]
and
\[
M(\xi - e^k < x) = \begin{cases}
\mu(x + \sqrt{k}/2), & \text{if } x \leq (x_0 - e^k) \\
1 - \mu(x + \sqrt{k}/2), & \text{if } x \geq (x_0 - e^k)
\end{cases}
\]
for any real number \( x \).
If \( x_0 - e^k \geq 0 \), then
\[
E(\xi - e^k) = \int_{0}^{+\infty} \left( x - e^k \right)^2 \mu(x + \sqrt{k}/2) dx
\]
\[
+ \int_{-\infty}^{x_0} \left( x - e^k \right)^2 \mu(x + \sqrt{k}/2) dx
\]
\[
= (x_0 - e^k)^2 + \frac{1}{2} \int_{x_0}^{+\infty} (x - e^k)^2 \mu(x) dx
\]
\[
+ \frac{1}{2} \int_{-\infty}^{x_0} (x - e^k)^2 \mu(x) dx
\]
\[
= (x_0 - e^k)^2 + \frac{1}{2} \int_{x_0}^{+\infty} (x - e^k)^2 \mu(x) dx
\]
\[
+ \frac{1}{2} \int_{-\infty}^{x_0} (x - e^k)^2 \mu(x) dx.
\]
If $x_0 - \epsilon < 0$, then
\[
E(\xi - e^3) = \int_0^{x_0 - \epsilon} M(x - e^3 < x)d\mu x - \int_{x_0 - \epsilon}^{\infty} M(x - e^3 < x)d\mu x - \int_{x_0 - \epsilon}^{\infty} \frac{\mu(x) + \varphi(x)}{2} dx
\]
\[
= \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \frac{\varphi(x)}{2} dx - \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \mu(x) dx
\]
\[
= \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \mu(x) dx - \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \varphi(x) dx
\]
\[
(\xi - e^3) + \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \mu(x) dx - \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} \varphi(x) dx
\]
\[
= (\xi - e^3) + \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} (x - e^3) \mu(x) dx - \frac{1}{2} \int_{x_0 - \epsilon}^{\infty} (x - e^3) \varphi(x) dx
\]
\[
= (\xi - e^3) + \frac{k}{2} \int_{x_0 - \epsilon}^{\infty} (x - e^3) \mu(x) dx
\]
\[
E(\xi - e^3) = \frac{k}{2} \int_{x_0 - \epsilon}^{\infty} (x - e^3) \mu(x) dx.
\]

The theorem is thus proved. $\square$

Example 4.1. Let $k$ be an even number, and let $\xi$ be an uncertain set with finite expected value $e$. If membership function $\mu$ is regular and symmetric, then the $k$-th moment of $\xi$ is
\[
E(\xi - e^k) = \frac{k}{2} \int_{-\infty}^{+\infty} (x - e^k) \mu(x) dx.
\]

Example 4.2. Consider the 1-st and 2-nd central moments of uncertain set with membership $\mu$ and finite expected value $e$. From the equation (9) and Theorem 2.3, we have
\[
E(\xi - e^k) = 0.
\]

From the equation (8), we have
\[
E(\xi - e^2) = \int_{-\infty}^{+\infty} \left( \sup_{x_1 \in X} \mu(x_1) + \sup_{x_2 \in X} \mu(x_2) \right) dx
\]
\[
= V(\xi).
\]

Here it is shown that the 1-st central moment of uncertain set is identically vanishing and the 2-nd central moment of uncertain set is just its variance.

5. Conclusion

This paper mainly studied the moments and central moments of uncertain set. At the same time, some formulas were given to calculate the moments and central moments of an uncertain set using the membership function. In addition, some examples were provided to calculate the moments and central moments of uncertain set.

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References