The stability of multifactor uncertain differential equation

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Abstract. This paper focuses on the stability of multifactor uncertain differential equation. The stability for the solution of multifactor uncertain differential equation is investigated, including stability in measure and stability in mean. Some stability theorems for the solution of such type of uncertain differential equation are given, in which some sufficient conditions for a multifactor uncertain differential equation being stable are provided. In addition, the relationship between stability in measure and stability in mean is discussed in this paper.

Keywords: Uncertainty theory, uncertain differential equation, stability

1. Introduction

People's decision making is usually influenced by the belief degrees. Especially, in the case of having no enough samples about some event, we always invite some domain experts to give the degrees of belief that the event will occur based on their knowledge, and those belief degrees will be as a basis to make decisions. Some people may think that the belief degrees can be treated as subjective probability. However, Liu \cite{Li1} showed that it is inappropriate to model belief degrees by using probability theory because it may lead to counterintuitive results.

For rationally dealing with belief degrees, Liu \cite{Li7} founded uncertainty theory in 2007, which is different from probability theory. Probability theory is mainly to deal with frequencies, and uncertainty theory is mainly to deal with belief degrees associated with human uncertainty. Uncertainty theory is based on the normality, duality, subadditivity and product axioms of uncertain measure. To represent quantities with uncertainty, the concept of uncertain variable was proposed by Liu \cite{Li7}. Furthermore, for modeling the evolution of uncertain phenomena, Liu \cite{Li8} introduced the concept of uncertain process in 2008 and obtained the theorems on the uncertainty distribution of first hitting time and extreme value for sample-continuous independent increment processes. Especially, a canonical Liu process was firstly investigated by Liu \cite{Li9}, which is a type of stationary independent increment process whose increments are normal uncertain variables, and a theory of uncertain calculus with respect to canonical Liu process was established by Liu.

Uncertain differential equation is a type of differential equation driven by canonical Liu process which was proposed by Liu \cite{Li8} in 2008. Since it plays an important role to deal with dynamical systems with uncertainty, it was studied by many researchers, and a lot of results in both theory and practice have been received. Chen and Liu \cite{CL} proved the existence and uniqueness theorem of solution of uncertain

Since Liu [9] firstly proposed an uncertain stock model, uncertain differential equations were applied successfully to financial fields. Liu [9], Chen [2] and Zhang and Liu [23] presented the pricing formulas of European option, American option and geometric average Asian option for Liu’s uncertain stock model, respectively. Peng and Yao [16] investigated the pricing option under the uncertain mean reverting stock model. The option pricing in the case of stock with dividends in uncertain financial market was studied by Chen, Liu and Ralescu [4]. Liu, Chen and Ralescu [15] introduced the uncertain currency model and currency option pricing method. Chen and Gao [3] proposed uncertain interest rate models, and Zhang, Ralescu and Liu [24] explored the interest rate option pricing. Besides, uncertain differential equation has also been applied to uncertain optimal control (Zhu [25]), and uncertain differential game (Yang and Gao [18]), and so on.

Considering the case of the uncertain factor influencing dynamic systems is usually not alone, Liu and Yao [12] discussed the uncertain integral with respect to multiple canonical Liu processes. Li, Peng and Zhang [6] proposed a type of multifactor uncertain differential equation, and proved that the multifactor uncertain differential equation has a unique solution under the Lipschitz condition and linear growth condition. In this paper, we will discuss the stability of the solution for this type of multifactor uncertain differential equation, and give some stability theorems for it.

The rest of this paper is organized as follows. In next section, some preliminary knowledge of uncertainty theory is introduced. In Section 3 and 4, the concepts of stability in measure and stability in mean for multifactor uncertain differential equation are presented, and some stability theorems are proved. In Section 5, the comparison of stability in measure and stability in mean is given. In Section 6, an example of applications of multifactor uncertain differential equation in population dynamics is presented. Finally, a brief summary is contained in Section 7.

2. Preliminaries

In this section, some preliminaries from uncertainty theory as needed are reviewed for further understanding the paper.

Definition 2.1. (Liu [7]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( M : \mathcal{L} \to [0, 1] \) is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) \( M(\Gamma) = 1 \) for the universal set \( \Gamma \);

Axiom 2. (Duality Axiom) \( M(\Lambda) + M(\Lambda^c) = 1 \) for any event \( \Lambda \);

Axiom 3. (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
M \left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i). \tag{2.1}
\]

A set \( \Lambda \in \mathcal{L} \) is called an event. The uncertain measure \( M(\Lambda) \) indicates the degree of belief that \( \Lambda \) will occur. The triplet \( (\Gamma, \mathcal{L}, M) \) is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [9].

Axiom 4. (Product Axiom) Let \( (\Gamma_k, \mathcal{L}_k, M_k) \) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( M \) is an uncertain measure satisfying

\[
M \left( \prod_{k=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} M_k(\Lambda_k) \tag{2.2}
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \ldots \), respectively.

Definition 2.2. (Liu [7]) An uncertain variable is a function from an uncertainty space \( (\Gamma, \mathcal{L}, M) \) to the set of real numbers, such that for any Borel set \( B \) of real numbers, the set
\[ \{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \} \quad (2.3) \]

is an event.

**Definition 2.3.** (Liu [10]) The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by
\[
\Phi(x) = M(\xi \leq x) \quad (2.4)
\]
for any real number \( x \).

**Definition 2.4.** (Liu [7]) An uncertain variable \( \xi \) is called normal if it has a normal uncertainty distribution
\[
\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e - x)}{\sqrt{3} \sigma} \right) \right)^{-1} \quad (2.5)
\]
denoted by \( N(e, \sigma) \) where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \).

**Definition 2.5.** (Liu [10]) An uncertainty distribution \( \Phi(x) \) is said to be regular if it is a continuous and strictly increasing function with respect to \( x \) at which 
\[
0 < \Phi(x) < 1, \quad \text{and} \quad \lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1. \quad (2.6)
\]

**Definition 2.6.** (Liu [10]) Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi(x) \). Then the inverse function \( \Phi^{-1}(\alpha) \) is called the inverse uncertainty distribution of \( \xi \).

**Definition 2.7.** (Liu [7]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by
\[
E[\xi] = \int_{0}^{+\infty} N(\xi \geq r)dr - \int_{-\infty}^{0} N(\xi \leq r)dr \quad (2.7)
\]
provided that at least one of the two integrals is finite.

**Theorem 2.1.** (Liu [7]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then
\[
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx. \quad (2.8)
\]

**Theorem 2.2.** (Liu [10]) Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). Then
\[
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha. \quad (2.9)
\]

An uncertain process is a sequence of uncertain variables indexed by a totally ordered set \( T \). A formal definition is given below.

**Definition 2.8.** (Liu [8]) Let \( (\Gamma, \mathcal{L}, M) \) be an uncertainty space and let \( T \) be a totally ordered set (e.g. time). An uncertain process is a function \( X_t(\gamma) \) from \( T \times (\Gamma, \mathcal{L}, M) \) to the set of real numbers such that \( \{ X_t \in B \} \) is an event for any Borel set \( B \) at each time \( t \).

**Definition 2.9.** (Liu [9]) An uncertain process \( C_t \) is said to be a canonical Liu process if
(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous;
(ii) \( C_t \) has stationary and independent increments;
(iii) every increment \( C_{s+t} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \).

In order to deal with the integration and differentiation of uncertain processes, Liu [9] proposed an uncertain integral with respect to canonical Liu process.

**Definition 2.10.** (Liu [9]) Let \( X_t \) be an uncertain process and \( C_t \) be a canonical Liu process. For any partition of closed interval \( [a, b] \) with \( a = t_1 < t_2 < \cdots < t_{k+1} = b \), the mesh is defined as
\[
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|. \quad (2.10)
\]
Then the Liu integral of \( X_t \) is defined as
\[
\int_{a}^{b} X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{\Delta} X_{t_i}(C_{t_{i+1}} - C_{t_i}) \quad (2.11)
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process \( X_t \) is said to be Liu integrable.

**Definition 2.11.** (Chen and Ralescu [5]) Let \( C_t \) be a canonical Liu process and let \( Z_t \) be an uncertain process. If there exist uncertain processes \( \mu_t \) and \( \sigma_t \) such that
\[
Z_t = Z_0 + \int_{0}^{t} \mu_s ds + \int_{0}^{t} \sigma_s dC_s \quad (2.12)
\]
for any \( t \geq 0 \), then \( Z_t \) is called a Liu process with drift \( \mu_t \) and diffusion \( \sigma_t \). Furthermore, \( Z_t \) has an uncertain differential
\[
dZ_t = \mu_t dt + \sigma_t dC_t. \quad (2.13)
\]
Liu [9] verified the fundamental theorem of uncertain calculus, i.e., for a canonical Liu process \( C_t \) and a
Definition 2.12. (Liu [8]) Suppose \( C_t \) is a canonical Liu process, and \( f \) and \( g \) are two functions. Then
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t
\] is called an uncertain differential equation.

Definition 2.13. (Liu and Yao [12])
Let \( C_{1t}, C_{2t}, \ldots, C_{nt} \) be independent canonical Liu processes and \( Z_t \) be an uncertain process. If there exist uncertain processes \( \mu_t \) and \( \sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt} \) such that
\[
Z_t = Z_0 + \int_0^t \mu_s ds + \sum_{i=1}^n \int_0^t \sigma_{is} dC_{is} \tag{2.16}
\]
for any \( t \geq 0 \), then we say \( Z_t \) has an uncertain differential
\[
dZ_t = \mu_t dt + \sum_{i=1}^n \sigma_{it} dC_{it}. \tag{2.17}
\]
In this case, \( Z_t \) is called a differentiable uncertain process with drift \( \mu_t \) and diffusions \( \sigma_{1t}, \sigma_{2t}, \ldots, \sigma_{nt} \).

Theorem 2.3. (Yao, Gao and Gao [21]) Let \( C_t \) be a canonical Liu process. Then there exists an uncertain variable \( K \) such that for each \( \gamma, K(\gamma) \) is a Lipschitz constant of the sample path \( C_t(\gamma) \),
\[
\lim_{x \to +\infty} \mathcal{M}\{K \leq x\} = 1 \tag{2.18}
\]
and
\[
\mathcal{M}\{\gamma \in \Gamma|K(\gamma) \leq x\} \geq 2 \left( 1 + \exp \left( \frac{-\pi x}{\sqrt{3}} \right) \right)^{-1} - 1. \tag{2.19}
\]

Theorem 2.4. (Liu [7]) Let \( \xi \) be an uncertain variable. Then for any given number \( r > 0 \), we have
\[
\mathcal{M}\{|\xi| \geq r\} \leq \frac{E[|\xi|]}{r}. \tag{2.20}
\]

3. Stability in measure

Considering the case of uncertain factor influencing dynamic systems is usually not alone, Liu and Yao [12] proposed uncertain integral with respect to multiple canonical Liu processes. Li, Peng and Zhang [6] presented a type of uncertain differential equation driven by multiple canonical Liu processes which is called a multifactor uncertain differential equation, and they proved the existence and uniqueness theorem for its solution. In this section, we will discuss the stability in measure for this type of uncertain differential equation.

Definition 3.1. (Li, Peng and Zhang [6])
Let \( C_{1t}, C_{2t}, \ldots, C_{nt} \) be independent canonical Liu processes and \( f, g_1, g_2, \ldots, g_n \) be some given functions. Then
\[
dX_t = f(t, X_t)dt + \sum_{i=1}^n g_i(t, X_t)dC_{it} \tag{3.1}
\]
is called a multifactor uncertain differential equation with respect to \( C_{1t}, C_{2t}, \ldots, C_{nt} \). A solution is an uncertain process \( X_t \) that satisfies (3.1) identically at each time \( t \).

The uncertain differential equation (3.1) is equivalent to the uncertain integral equation
\[
X_t = X_0 + \int_0^t f(s, X_s) ds + \sum_{i=1}^n \int_0^t g_i(s, X_s) dC_{is}. \tag{3.2}
\]

Definition 3.2. A multifactor uncertain differential equation
\[
dX_t = f(t, X_t)dt + \sum_{i=1}^n g_i(t, X_t)dC_{it}
\]
is said to be stable in measure if for any two solutions \( X_t \) and \( Y_t \) with different initial values \( X_0 \) and \( Y_0 \), we have
\[
\lim_{|X_0-Y_0| \to 0} \mathcal{M}\left\{ \sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon \right\} = 1 \tag{3.3}
\]
for any given number \( \varepsilon > 0 \).

Theorem 3.1. Assume the uncertain differential equation
\[
dX_t = f(t, X_t)dt + \sum_{i=1}^n g_i(t, X_t)dC_{it}
\]
has a unique solution for each given initial value. Then it is stable in measure if the coefficients $f(t, x)$ and $g_i(t, x), g_2(t, x), \ldots, g_n(t, x)$ satisfy the strong Lipschitz condition

$$|f(t, x) - f(t, y)| + \sum_{i=1}^{n} |g_i(t, x) - g_i(t, y)| \leq L_t|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0$$

where $L_t$ is some positive function satisfying

$$\int_{0}^{+\infty} L_t dt < +\infty. \quad (3.5)$$

**Proof.** Let $X_t$ and $Y_t$ be the solutions of the multifactor uncertain differential equation (3.1) with different initial values $X_0$ and $Y_0$, respectively. Then for Lipschitz continuous sample paths $C_i(t), i = 1, 2, \ldots, n$, we have

$$X_t(\gamma) = X_0 + \int_{0}^{t} f(s, X_s(\gamma)) ds + \sum_{i=1}^{n} \int_{0}^{t} g_i(s, X_s(\gamma)) dC_i(s(\gamma)) \quad (3.6)$$

and

$$Y_t(\gamma) = Y_0 + \int_{0}^{t} f(s, Y_s(\gamma)) ds + \sum_{i=1}^{n} \int_{0}^{t} g_i(s, Y_s(\gamma)) dC_i(s(\gamma)). \quad (3.7)$$

By the strong Lipschitz condition, we have

$$|X_t(\gamma) - Y_t(\gamma)| \leq |X_0 - Y_0| + \int_{0}^{t} |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| ds \\
+ \sum_{i=1}^{n} \int_{0}^{t} |g_i(s, X_s(\gamma)) - g_i(s, Y_s(\gamma))| |dC_i(s(\gamma))| \leq |X_0 - Y_0| + \int_{0}^{t} L_s|X_s(\gamma) - Y_s(\gamma)| ds \\
+ \int_{0}^{t} K(\gamma)L_s|X_s(\gamma) - Y_s(\gamma)| ds = |X_0 - Y_0| + (1 + K(\gamma)) \int_{0}^{t} L_s|X_s(\gamma) - Y_s(\gamma)| ds \quad (3.8)$$

where $K(\gamma) = \sum_{i=1}^{n} K_i(\gamma)$ and $K_i(\gamma)$ are the Lipschitz constants of $C_i(\gamma), i = 1, 2, \ldots, n$, respectively. It follows from the Gronwall’s inequality that

$$|X_t(\gamma) - Y_t(\gamma)| \leq |X_0 - Y_0| \exp\left((1 + K(\gamma)) \int_{0}^{t} L_s ds\right) \quad (3.9)$$

for any $t \geq 0$. So

$$\sup_{t \geq 0} |X_t(\gamma) - Y_t(\gamma)| \leq |X_0 - Y_0| \exp\left((1 + K(\gamma)) \int_{0}^{+\infty} L_s ds\right) \quad (3.10)$$

almost surely, where $K$ is a nonnegative uncertain variable such that

$$\lim_{x \to +\infty} \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} = 1 \quad (3.11)$$

by Theorem 2.3. Then there exists a real number $H$ such that

$$\mathcal{M}\{\gamma | K(\gamma) \leq H\} \geq 1 - \varepsilon \quad (3.12)$$

for any given $\varepsilon > 0$. We take

$$\delta = \exp\left(-\left(1 + H\right) \int_{0}^{+\infty} L_s ds\right) \varepsilon. \quad (3.13)$$

Then we have $|X_t(\gamma) - Y_t(\gamma)| \leq \varepsilon, \forall t \geq 0$ provided that $|X_0 - Y_0| \leq \delta$ and $K(\gamma) \leq H$. So if $|X_0 - Y_0| \leq \delta$ we have

$$\mathcal{M}\left\{\sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon\right\} > 1 - \varepsilon. \quad (3.14)$$

It means that

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\left\{\sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon\right\} = 1. \quad (3.15)$$

Therefore the multifactor uncertain differential equation (3.1) is stable in measure. □

**Remark 3.1.** Theorem 3.1 gives the sufficient condition but not the necessary condition for multifactor uncertain differential equation being stable in measure.

**Example 3.1.** Consider the multifactor uncertain differential equation

$$dX_t = \mu dt + \sigma_1 dC_{1t} + \sigma_2 dC_{2t}. \quad (3.16)$$
Since its solutions with different initial values \( X_0 \) and \( Y_0 \) are
\[
X_t = X_0 + \mu t + \sigma_1 C_{1t} + \sigma_2 C_{2t},
\]
\[
Y_t = Y_0 + \mu t + \sigma_1 C_{1t} + \sigma_2 C_{2t},
\]
respectively, we have
\[
\sup_{t \geq 0} |X_t - Y_t| = |X_0 - Y_0| \tag{3.18}
\]
almost surely. Then
\[
\lim_{|X_0 - Y_0| \to 0} \mathcal{M} \left\{ \sup_{t \geq 0} |X_t - Y_t| \leq \varepsilon \right\} \leq \lim_{|X_0 - Y_0| \to 0} \mathcal{M} \{ |X_0 - Y_0| \leq \varepsilon \} = 1. \tag{3.19}
\]
Therefore the multifactor uncertain differential equation (3.16) is stable in measure.

4. Stability in mean

In this section, we investigate the stability in mean for multifactor uncertain differential equation.

**Definition 4.1.** A multifactor uncertain differential equation
\[
dX_t = f(t, X_t) + \sum_{i=1}^{n} g_i(t, X_t) dC_{it} \tag{4.1}
\]
is said to be stable in mean if for any two solutions \( X_t \) and \( Y_t \) with different initial values \( X_0 \) and \( Y_0 \), we have
\[
\lim_{|X_0 - Y_0| \to 0} E \left[ \sup_{t \geq 0} |X_t - Y_t| \right] = 0. \tag{4.2}
\]

**Theorem 4.1.** The multifactor uncertain differential equation (4.1) is stable in mean if the coefficients \( f(t, x) \) and \( g_1(t, x), g_2(t, x), \ldots, g_n(t, x) \) satisfy the strong Lipschitz condition
\[
|f(t, x) - f(t, y)| \leq L_{1t} |x - y|,
\]
\[
\sum_{i=1}^{n} |g_i(t, x) - g_i(t, y)| \leq L_{2t} |x - y|, \tag{4.3}
\]
\[
\forall x, y \in \mathbb{R}, t \geq 0
\]
where \( L_{1t} \) and \( L_{2t} \) are two functions satisfying
\[
\int_{0}^{+\infty} L_{1t} dt < +\infty, \quad \int_{0}^{+\infty} L_{2t} dt < \frac{\pi}{\sqrt{3}} \tag{4.4}
\]

**Proof.** Let \( X_t \) and \( Y_t \) be the solutions of the multifactor uncertain differential equation (4.1) with different initial values \( X_0 \) and \( Y_0 \), respectively. Then for Lipschitz continuous sample paths \( C_i(\gamma), i = 1, 2, \ldots, n \), we have
\[
X_t = X_0 + \int_{0}^{t} f(s, X_s(\gamma)) ds + \sum_{i=1}^{n} \int_{0}^{t} g_i(s, X_s(\gamma)) dC_{is}(\gamma)
\]
\[
y = Y_0 + \int_{0}^{t} f(s, Y_s(\gamma)) ds + \sum_{i=1}^{n} \int_{0}^{t} g_i(s, Y_s(\gamma)) dC_{is}(\gamma).
\]
By the strong Lipschitz condition, we have
\[
|X_t(\gamma) - Y_t(\gamma)| \
\leq |X_0 - Y_0| + \int_{0}^{t} |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| ds + \sum_{i=1}^{n} \int_{0}^{t} |g_i(s, X_s(\gamma)) - g_i(s, Y_s(\gamma))| |dC_{is}(\gamma)|
\]
\[
\leq |X_0 - Y_0| + \int_{0}^{t} L_{1s} |X_s(\gamma) - Y_s(\gamma)| ds + \sum_{i=1}^{n} \int_{0}^{t} K(\gamma) L_{2s} |X_s(\gamma) - Y_s(\gamma)| ds
\]
\[
= |X_0 - Y_0| + \int_{0}^{t} (L_{1s} + K(\gamma) L_{2s}) |X_s(\gamma) - Y_s(\gamma)| ds \tag{4.7}
\]
where \( K(\gamma) = \sqrt{\frac{n}{1}} K_i(\gamma) \) and \( K_i(\gamma) \) are the Lipschitz constants of \( C_i(\gamma), i = 1, 2, \ldots, n \), respectively, and \( K_i(\gamma), i = 1, 2, \ldots, n \) are independent. It follows from the Gronwall’s inequality that
\[
|X_t(\gamma) - Y_t(\gamma)| \
\leq |X_0 - Y_0| \exp \left( \int_{0}^{t} L_{1s} ds \right) \left( \exp \left( \int_{0}^{t} K(\gamma) L_{2s} ds \right) \right) \tag{4.8}
\]
\[
\leq |X_0 - Y_0| \exp \left( \int_{0}^{+\infty} L_{1s} ds \right) \left( \exp \left( \int_{0}^{+\infty} K(\gamma) L_{2s} ds \right) \right)
\]
for any $t \geq 0$. Then we have

$$
\sup_{t \geq 0} |X_t - Y_t| 
\leq |X_0 - Y_0| \exp \left( \int_0^{+\infty} L_{1x} ds \right) \exp \left( K \int_0^{+\infty} L_{2x} ds \right)
$$

(4.9)

almost surely, where $K$ is a nonnegative uncertain variable. Since $\{K(\gamma) \leq x\} \Leftrightarrow \left\{ \bigcap_{i=1}^n K_i(\gamma) \leq x \right\}$, we have

$$
\mathcal{M} \{ \gamma | K(\gamma) \leq x \} \geq 2 \left( 1 + \exp \left( \frac{-\pi x}{\sqrt{3}} \right) \right) - 1
$$

(4.10)

by Theorem 2.3 and the independence of $K_i(\gamma), i = 1, 2, \ldots, n$. Taking expected value on both sides of (4.9), we have

$$
E \left[ \sup_{t \geq 0} |X_t - Y_t| \right] 
\leq |X_0 - Y_0| \exp \left( \int_0^{+\infty} L_{1x} ds \right) E \left[ \exp \left( K \int_0^{+\infty} L_{2x} ds \right) \right].
$$

(4.11)

Noticing

$$
\int_0^{+\infty} L_{1x} ds < +\infty,
$$

(4.12)

we have

$$
\exp \left( \int_0^{+\infty} L_{1x} ds \right) < +\infty,
$$

(4.13)

and since

$$
\int_0^{+\infty} L_{2x} ds < \frac{\pi}{\sqrt{3}},
$$

(4.14)

we have

$$
E \left[ \exp \left( K \int_0^{+\infty} L_{2x} ds \right) \right] 
= \int_0^{+\infty} \mathcal{M} \left\{ \exp \left( K \int_0^{+\infty} L_{2x} ds \right) \geq x \right\} dx
\leq 1 + \int_0^{+\infty} \mathcal{M} \left\{ \exp \left( K \int_0^{+\infty} L_{2x} ds \right) \geq x \right\} dx
= 1 + \left( \int_0^{+\infty} L_{2x} ds \right) \int_0^{+\infty} \exp \left( y \int_0^{+\infty} L_{2x} ds \right) \mathcal{M} \{ K \geq y \} dy
\leq 1 + \left( \int_0^{+\infty} L_{2x} ds \right) \int_0^{+\infty} \exp \left( y \int_0^{+\infty} L_{2x} ds \right) \left( 1 - \left( 1 + \exp \left( \frac{-\pi y}{\sqrt{3}} \right) \right) - 1 \right) dy
= 1 + 2 \left( \int_0^{+\infty} L_{2x} ds \right) \int_0^{+\infty} \exp \left( y \int_0^{+\infty} L_{2x} ds \right) \left( 1 + \exp \left( \frac{\pi y}{\sqrt{3}} \right) \right) dy
\leq 1 + 2 \int_1^{+\infty} \frac{1}{1 + x} \mathcal{M} \{ K \geq \gamma \} \mathcal{M} \{ \gamma | K(\gamma) \leq x \}\mathcal{M} \{ K \geq y \} dy
< +\infty.
$$

(4.15)

It follows from the definition of stability in mean and (4.11), (4.13) and (4.15) that the multifactor uncertain differential equation (4.1) is stable in mean under the strong Lipschitz condition.

\[ \square \]

**Remark 4.1.** Theorem 4.1 gives the sufficient condition but not the necessary condition for multifactor uncertain differential equation being stable in mean.

**Example 4.1.** Consider the multifactor uncertain differential equation

$$
dX_t = \mu dt + \sigma_1 dC_{1t} + \sigma_2 dC_{2t}.
$$

(4.16)

in the Example 3.1.

It follows from the discussion in the Example 3.1, we have

$$
\sup_{t \geq 0} |X_t - Y_t| = |X_0 - Y_0|
$$

(4.17)

almost surely. Then

$$
\lim_{|X_0 - Y_0| \to 0} E \left\{ \sup_{t \geq 0} |X_t - Y_t| \right\}
= \lim_{|X_0 - Y_0| \to 0} E \left| X_0 - Y_0 \right|
= 0.
$$

(4.18)
Therefore the multifactor uncertain differential equation (4.16) is stable in mean.

**Example 4.2.** Consider the multifactor uncertain differential equation

\[ dX_t = X_t \, dt + \sigma_1 \, dC_{1t} + \sigma_2 \, dC_{2t}. \]  \hspace{1cm} (4.19)

Since its solutions with different initial values \( X_0 \) and \( Y_0 \) are

\[ X_t = \exp(t)X_0 + \sum_{i=1}^{2} \sigma_i \exp(t) \int_0^t \exp(-s) \, dC_{is}, \]
\[ Y_t = \exp(t)Y_0 + \sum_{i=1}^{2} \sigma_i \exp(t) \int_0^t \exp(-s) \, dC_{is}, \]

respectively, we have

\[ \sup_{t \geq 0} |X_t - Y_t| = \sup_{t \geq 0} |X_0 - Y_0| \exp(t) = +\infty \]

almost surely. Then

\[ \lim_{|X_0 - Y_0| \to 0} E \left( \sup_{t \geq 0} |X_t - Y_t| \right) = \lim_{|X_0 - Y_0| \to 0} E \left[ |X_0 - Y_0| \exp(t) \right] = +\infty. \]  \hspace{1cm} (4.22)

Therefore the multifactor uncertain differential equation (4.19) is not stable in mean.

5. Comparison of stability in measure and stability in mean

In this section, the relationship between stability in measure and stability in mean for a multifactor uncertain differential equation is discussed.

**Theorem 5.1.** For a multifactor uncertain differential equation, if it is stable in mean, then it is stable in measure.

**Proof.** Suppose that \( X_t \) and \( Y_t \) are two solutions of a multifactor uncertain differential equation with different initial values \( X_0 \) and \( Y_0 \), respectively. Then it follows from the Definition 4.1 of stability in mean that

\[ \lim_{|X_0 - Y_0| \to 0} E \left( \sup_{t \geq 0} |X_t - Y_t| \right) = 0. \]  \hspace{1cm} (5.1)

By using Theorem 2.4, for any given number \( \varepsilon > 0 \), we have

\[ \lim_{|X_0 - Y_0| \to 0} \mathbb{M} \left[ \sup_{t \geq 0} |X_t - Y_t| \geq \varepsilon \right] \leq \lim_{|X_0 - Y_0| \to 0} \mathbb{E} \left[ \sup_{t \geq 0} |X_t - Y_t| \right] / \varepsilon = 0. \]  \hspace{1cm} (5.2)

Therefore it follows from the Definition 3.2 of stability in measure that the multifactor uncertain differential equation is stable in measure.

**Remark 5.1.** For a multifactor uncertain differential equation, generally, stability in measure does not imply stability in mean.

**Example 5.1.** Consider the multifactor uncertain differential equation

\[ dX_t = \frac{2(X_t - C_{2t})}{(1 + t)^2} \, dC_{1t} + dC_{2t}. \]  \hspace{1cm} (5.3)

As we can see, the coefficients \( f(t, x) = 0, \ g_1(t, x) = \frac{2(x - C_{2t})}{(1 + t)^2} \) and \( g_2(t, x) = 1 \) satisfy the strong Lipschitz condition in Theorem 3.1, so the multifactor uncertain differential equation (5.3) is stable in measure.

We can find that the equation (5.3) has a solution \( X_t = C_{2t} \) \hspace{1cm} (5.4)

with an initial value \( X_0 = 0 \) and a solution

\[ Y_t = Y_0 \exp \left( \int_0^t \frac{2}{(1 + s)^2} \, dC_{1s} \right) + C_{2t}, \]  \hspace{1cm} (5.5)

with an initial value \( Y_0 \neq 0 \). Then

\[ \sup_{t \geq 0} |X_t - Y_t| = |Y_0| \sup_{t \geq 0} \exp \left( \int_0^t \frac{2}{(1 + s)^2} \, dC_{1s} \right) \]  \hspace{1cm} (5.6)

almost surely, and

\[ E \left[ \sup_{t \geq 0} |X_t - Y_t| \right] = |Y_0| E \left[ \sup_{t \geq 0} \exp \left( \int_0^t \frac{2}{(1 + s)^2} \, dC_{1s} \right) \right] \]  \hspace{1cm} (5.7)

\[ \geq |Y_0| E \left[ \exp \left( \int_0^{+\infty} \frac{2}{(1 + s)^2} \, dC_{1s} \right) \right]. \]

Since

\[ \int_0^{+\infty} \frac{2}{(1 + s)^2} \, dC_{1s} \sim \mathcal{N} \left( 0, \int_0^{+\infty} \frac{2}{(1 + s)^2} \, ds \right) = \mathcal{N}(0, 2), \]  \hspace{1cm} (5.8)
expressed by the equation be the mortality rate. Then the rate of change can be
t the species at time P
where b where
in continuous time. Malthusian equation Malthusian growth as follows
differential equation model for population dynamics change, and so on. We present a multifactor uncertain
tain factors, such as resources, ecology environment the population is always influenced by various uncer-
tions of multifactor uncertain differential equation in
6. An example of applications in population dynamics

In this section, we present an example of applications of multifactor uncertain differential equation in population dynamics. Let P(t) be the population of the species at time t, b be the reproduction rate and d be the mortality rate. Then the rate of change can be expressed by the equation

\[
\frac{dP(t)}{dt} = bP(t) - dP(t). \tag{6.1}
\]

As we know, the above equation has a solution

\[
P(t) = P_0 \exp((b - d)t), \tag{6.2}
\]

where P_0 is the initial population. This is called Malthusian growth, and the equation (6.1) is called Malthusian equation in continuous time.

However, it is unrealistic that the reproduction rate and the mortality rate are regarded as constants, since the population is always influenced by various uncertain factors, such as resources, ecology environment change, and so on. We present a multifactor uncertain differential equation model for population dynamics as follows

\[
\frac{dP(t)}{dt} = \left( b + \sigma_1 \frac{dC_{1t}}{dt} \right) P(t) - \left( d + \sigma_2 \frac{dC_{2t}}{dt} \right) P(t), \tag{6.3}
\]

where b is the expected reproduction rate, \(\sigma_1\) is the volatility for the reproduction rate, d is the expected mortality rate and \(\sigma_2\) is the volatility for the mortality rate, C_{1t} and C_{2t} are the independent canonical Liu processes.

The equation (6.3) is equivalent to the following

\[
dP(t) = (b - d)P(t)dt + \sigma_1 P(t) dC_{1t} - \sigma_2 P(t) dC_{2t}. \tag{6.4}
\]

If b = d, then the equation (6.4) is stable in measure. Compared with ordinary differential equation, modeling the population dynamics by using multifactor uncertain differential equation model (6.4) is more appropriate since the uncertain fluctuation is considered.

7. Conclusion

Multifactor uncertain differential equation is a type of differential equation driven by multiple canonical Liu processes. The concepts of stability in measure and in mean for this type of uncertain differential equation were proposed in this paper, and some theorems on stability in measure and in mean were proved, in which the sufficient conditions the multifactor uncertain differential equation being stable were provided. In addition, the relationship between stability in measure and stability in mean were discussed.

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References