The bounds of premium and optimality of stop loss insurance under uncertain random environments

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A B S T R A C T
The potential loss of insured can be affected by many nondeterministic factors, in which uncertainty always coexists with randomness. Therefore, uncertain random variables are used to describe this phenomenon of simultaneous appearance of both uncertainty and randomness in potential loss. Based on that, the upper and lower bounds of premium with uncertain random loss are given, respectively. Moreover, a mathematical model of uncertain random optimal insurance problem is established and the stop loss insurance is proved to be the optimal insurance policy and the equation for calculating the optimal deductible is arrived. Some numerical examples are also given for illustration.

1. Introduction

The optimal insurance problem is about how a risk-averse insured balances the tradeoff and chooses his optimal insurance so as to make him well off in the future. For decades, there are substantial theoretical and empirical studies to determine the optimal form of insurance that is most preferred by the insureds. For instance, Borch (1962) characterized a Pareto optimal risk sharing between several risk-averse individuals. Under fairly general conditions, Arrow (1963) arrived at that the optimal insurance was deductible insurance. In 1974, Arrow (1974) showed that full coverage above a fixed deductible was optimal for a utility maximizing individual. Zhou and Wu (2008) imposed the insurer’s risk constraint on Arrow’s optimal insurance model. The purpose of Raviv (1979) was to explain the prevalence of several different insurance contracts observable in the real world. Huberman et al. (1983) considered two commonly observed insurance policy provisions: upper limits on coverage and deductibles. Touzi (2000) studied the stochastic control problem of maximizing expected utility from terminal wealth, when the wealth process was subject to shocks produced by a general marked point process. Pliska and Ye (2007) considered optimal insurance and consumption rules for a wage earner whose lifetime was random. Zhou et al. (2010) considered

the optimal insurance problem when the insurer had a loss limit constraint. Lu et al. (2012) derived an optimal insurance treaty when the insured faced multiple sources of risk. Golubin (2008) examined a classical risk model where both insurance and reinsurance policies were chosen by the insurer in order to minimize the expected maximal loss. Wang and Huang (2002) developed an optimal insurance contract endogenously and determined the optimal coverage levels with respect to deductible insurance, upper-limit insurance and proportional coinsurance. Kou and Cadenillas (2014) considered an insurer who wanted to maximize his/her expected utility of terminal wealth by selecting optimal investment and risk control strategies.

The indeterminacy of the risk in above contributions is considered as objective indeterminacy and treated by probability theory. However, sometimes indeterminacy shows its subjectivity, which is usually treated by fuzzy set theory in the past decades. Several researchers have applied fuzzy sets theory to optimal insurance problems recent years. For instance, in Cummins and Derrig (1993) considered fuzzy set theory as an effective method for combining statistical and judgmental criteria in actuarial decision making. Cummins and Derrig (1997) used fuzzy set theory to solve a problem in actuarial science, the financial pricing of property-liability insurance contracts. Young (1996) described how fuzzy logic could be used to make insurance pricing decisions that consistently considered supplementary data, including vague or linguistic objectives of the insurer. The specific purposes of Shapiro (2004) were two-fold: first, to review fuzzy logic applications in insurance so as to document the unique characteristics of insurance as an application area; and second, to document the extent to which fuzzy logic
technologies had been employed. Derrig and Ostaszewski (1999) presented the uses of fuzzy sets in areas such as: underwriting, risk classification, interest rates, rate making, valuation of premium and taxes. Li (2011) considered a risk-averse firm bearing the revenue risk and fuzzy production cost. Huang et al. (2011) formulated an actuarial model of life insurance for fuzzy process. Zeng (2011) built the fuzzy optimal model which was used for the risk orders of the insurance companies.

Although fuzzy theory had been widely applied, Liu (2012) showed that fuzzy variable was not a suitable tool for modeling some kinds of uncertain quantities. Liu (2007, p. 205–228) founded an uncertainty theory in 2007 and refined it in Liu (2010, p. 1–74). In recent years, uncertainty theory has been successfully employed for dealing with many uncertain situations, such as facility locations (Gao, 2012), optimal assignment problems (Zhang and Peng, 2013), maximum flow problems (Han et al., 2014), transportation problems (Yang et al., 2015) and so on.

In many cases, uncertainty and randomness simultaneously appear in a complex system. To solve this problem, Liu (2013a) introduced the concept of uncertain random variable and proposed the chance measure to evaluate the occurrence of an uncertain random event. Therefore, in this paper, an uncertain random variable is used to represent the potential loss of insured. The remainder of this paper is organized as follows. In Section 2, some important concepts of uncertainty theory and chance theory are introduced. Section 3 proposes the upper and lower bounds of premium with uncertain random loss, respectively. In Section 4, a mathematical model of uncertain random optimal insurance problem is established and the stop loss insurance is proved to be the optimal insurance policy and the equation for calculating the optimal deductible is arrived. Section 5 draws some conclusions.

2. Preliminaries

Let \( \Gamma \) be a nonempty set and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( A \in \mathcal{L} \) is called an event. Then a number \( \mathcal{M}(A) \) will be assigned to each event \( A \) to indicate the belief degree with which we believe \( A \) will happen. There is no doubt that the assignment is not arbitrary, and the uncertain measure \( \mathcal{M} \) must have certain mathematical properties. In order to rationally deal with belief degree, Liu (2007) suggested the following four axioms:

**Axiom 1 (Normality).** \( \mathcal{M}(\Gamma) = 1 \) for the universal set \( \Gamma \).

**Axiom 2 (Monotonicity).** \( \mathcal{M}(A_1) \leq \mathcal{M}(A_2) \) whenever \( A_1 \leq A_2 \).

**Axiom 3 (Duality).** \( \mathcal{M}(A) + \mathcal{M}(A^\complement) = 1 \) for any event \( A \).

**Axiom 4 (Countable Subadditivity).** For every countable sequence of events \( A_1, A_2, \ldots \), we have

\[
\mathcal{M}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(A_i).
\]

**Definition 1** (Liu, 2007). The set function \( \mathcal{M} \) is called an uncertain measure if it satisfies the four axioms.

**Definition 2** (Liu, 2007). Let \( \Gamma \) be a nonempty set, \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \), and \( \mathcal{M} \) an uncertain measure. Then the triplet \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is called an uncertainty space.

**Definition 3** (Liu, 2007). An uncertain variable is a function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers such that \( \{\xi \in \mathcal{B}\} \) is an event for any Borel set \( \mathcal{B} \).

**Definition 4** (Liu, 2007). The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by

\[
\Phi(x) = \mathcal{M}\{\xi \leq x\}
\]

for any real number \( x \).

**Definition 5** (Liu, 2010). An uncertain variable \( \xi \) is called zigzag if it has a zigzag uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < a \\
(x - a)/(b - a), & \text{if } a \leq x < b \\
(2c - x)/(2c - b), & \text{if } b \leq x < c \\
1, & \text{if } x \geq c
\end{cases}
\]

denoted by \( Z(a, b, c) \), where \( a, b, c \) are real numbers with \( a < b < c \).

**Definition 6** (Liu, 2013a). An uncertain random variable is a function \( \xi \) from a chance space \( (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P}) \) to the set of real numbers such that \( \{\xi \in \mathcal{B}\} \) is an event in \( \mathcal{L} \times \mathcal{A} \) for any Borel set \( \mathcal{B} \).

**Definition 7** (Liu, 2013a). Let \( \xi \) be an uncertain random variable on the chance space \( (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P}) \), and let \( \mathcal{B} \) be a Borel set. Then \( \{\xi \in \mathcal{B}\} \) is an uncertain random event with chance measure

\[
\text{Ch}[\xi \in \mathcal{B}] = \int_{\Omega} \mathcal{P}(\omega) \left[ \mathcal{M}\{y \in \Gamma \mid \xi(y, \omega) \in \mathcal{B}\} \geq x \right] dx.
\]

**Definition 8** (Liu, 2013a). Let \( \xi \) be an uncertain random variable. Then its chance distribution is defined by

\[
\Phi(x) = \text{Ch}[\xi \leq x]
\]

for any \( x \in \Re \).

**Theorem 1** (Liu, 2013b). Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent uncertain variables with uncertainty distribution \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. Then the uncertain random variable

\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)
\]

has a chance distribution

\[
\Phi(x) = \int_{\Omega_m} F(x; y_1, y_2, \ldots, y_m) d\Psi_1(y_1)d\Psi_2(y_2)\cdots d\Psi_m(y_m),
\]

where \( F(x; y_1, y_2, \ldots, y_m) \) is the uncertainty distribution of the uncertain variable

\[
f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)
\]

and is determined by its inverse function

\[
F^{-1}(\alpha; y_1, y_2, \ldots, y_m)
\]

provided that \( f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \) is a strictly increasing function with respect to \( \tau_1, \ldots, \tau_n \).

**Definition 9** (Liu, 2013a). Let \( \xi \) be an uncertain random variable. Then its expected value is defined by

\[
E[\xi] = \int_{\mathbb{R}_+} \text{Ch}[\xi \geq x] dx - \int_{\mathbb{R}_-} \text{Ch}[\xi \leq x] dx
\]

provided that at least one of the two integrals is finite.
Theorem 2 (Liu, 2010). Let \( \xi \) be an uncertain random variable with chance distribution \( \Phi \). Then
\[
E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x). \tag{10}
\]

3. The bounds on premium with uncertain random loss

In this section, we will discuss the upper and lower bounds of premium under uncertain random loss. A similar problem in stochastic case can refer to Kaas et al. (2008, p. 4), Xie and Han (2000, p. 198) or Chavas (2004, p. 33).

Suppose that an insured with initial wealth \( w \) has the utility function \( u_1(x) \). Consider the case of the insured facing a potential risk, as presented by an uncertain random variable \( X \), \( 0 \leq X \leq w \). Usually, the insured will consider whether to purchase an insurance contract in order to eliminate the risk. If he decides to buy an insurance contract with premium \( H \), he surely has property \( w - H \), his utility is \( u_1(w - H) \); and if he refuses to buy it, his property becomes an uncertain random variable \( -X \), whose utility is denoted by \( U_1(w - X) \). Therefore, the insurance premium \( H \) should satisfy
\[
u_1(w - H) \geq U_1(w - X), \tag{11}
\]
in which \( u_1(w - H) \) is the utility function of crisp number \( w - H \) and \( U_1(w - X) \) is the utility function of uncertain random variable \( w - X \). The insured hopes the premium \( H \) be as low as possible, so the upper bound of premium \( H^* \) is just the solution of the utility to buy an insurance contract is equal to the utility not to buy”, which implies that \( H^* \) is the solution of the implicit equation
\[
u_1(w - H) = E[U_1(w - X)]. \tag{12}
\]

On the other hand, the insurer is assumed to have an initial wealth \( v \) and his utility function is denoted by \( u_2(x) \). At first, his utility is \( u_2(v) \). If he decides to underwrite with premium \( G \), his property becomes an uncertain random variable \( v + G - X \), whose utility is denoted by \( U_2(v + G - X) \). From the insurer’s perspective, the reasonable premium \( G \) should satisfy
\[
u_2(v + G - X) \geq u_2(v), \tag{13}
\]
in which \( u_2(v + G - X) \) is the utility function of uncertain random variable \( v + G - X \) and \( u_2(v) \) is the utility function of crisp number \( v \). The insurer hopes the premium \( G \) be as high as possible, so the lower bound of premium \( G_* \) is the solution of “the utility to underwrite the insurance contract is equal to the utility not to underwrite", which implies that \( G_* \) is just the solution of the implicit equation
\[
E[U_2(v + G - X)] = u_2(v). \tag{14}
\]

If \( G_* > H^* \), the insurance contract cannot be sold; and if \( G_* < H^* \), the final decision of premium \( P \) should be in the interval \([G_* , H^*] \), which is called the feasible price area.

Example 1. Suppose that the utility functions of insured and insurer are \( u_1(x) = -x^2 + 800x \) and \( u_2(x) = x + 100 \), respectively. The initial wealth of the insured and insurer are 800 dollars and 1000 dollars, respectively. It is assumed that the insured’s loss of property includes both direct loss and indirect loss, in which direct loss \( \eta \) is uniformly distributed random variable \( U(200, 300) \) and indirect loss \( \tau \) is zigzag uncertain variable \( Z(100, 300, 400) \). So the total loss of insured is \( X = \eta + \tau \), which is considered as an uncertain random variable. The question is how to determine the feasible price area.

To solve this problem, at first, we need to calculate the chance distribution of \( X \). The probability distribution of \( \eta \) and the uncertainty distribution of \( \tau \) are denoted by \( \Phi(y) \) and \( \Upsilon(z) \), respectively. Then we obtain
\[
\Phi(y) = \begin{cases} 
0, & \text{if } y \leq 200 \\
\frac{y - 200}{400}, & \text{if } 200 \leq y \leq 300 \\
1, & \text{if } y \geq 300
\end{cases} \tag{15}
\]
and
\[
\Upsilon(z) = \begin{cases} 
0, & \text{if } z \leq 100 \\
\frac{1}{400} (z - 100), & \text{if } 100 \leq z \leq 300 \\
\frac{1}{200} (z - 200), & \text{if } 300 \leq z \leq 400 \\
1, & \text{if } z \geq 400.
\end{cases} \tag{16}
\]
By Theorem 1, we can arrive at the chance distribution of \( X \), namely,
\[
\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y) d\Phi(y) = \frac{1}{100} \int_{200}^{300} \Upsilon(x - y) dy. \tag{17}
\]

Denote \( u = x - y \), then
\[
\Phi(x) = \frac{1}{100} \int_{x - 300}^{x - 200} \Upsilon(u) du. \tag{18}
\]
When \( x \leq 300 \), we have
\[
\Phi(x) = 0. \tag{19}
\]
When \( 300 \leq x \leq 400 \), we have
\[
\Phi(x) = \frac{1}{100} \int_{x - 300}^{x - 200} \frac{1}{400} (u - 100) du = \frac{1}{80000} (x - 300)^2. \tag{20}
\]
When \( 400 \leq x \leq 500 \), we have
\[
\Phi(x) = \frac{1}{100} \int_{x - 300}^{x - 200} \frac{u - 100}{400} du = \frac{1}{800} (2x - 700). \tag{21}
\]
When \( 500 \leq x \leq 600 \), we have
\[
\Phi(x) = \frac{1}{100} \int_{x - 300}^{300} \frac{1}{400} (u - 100) du + \frac{1}{100} \int_{300}^{x - 200} \frac{1}{200} (u - 200) du
= \frac{1}{80000} (x^2 - 800x + 180000). \tag{22}
\]
When \( 600 \leq x \leq 700 \), we have
\[
\Phi(x) = \frac{1}{100} \int_{x - 300}^{400} \frac{1}{200} (u - 200) du + \frac{1}{100} \int_{400}^{x - 200} du
= \frac{1}{40000} (-x^2 + 1400x - 450000). \tag{23}
\]
When $x \geq 700$, we have
\[
\Phi(x) = 1. \tag{24}
\]

In summary, the chance distribution of the uncertain random variable $X$ is
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq 300 \\
\frac{1}{80000} (x - 300)^2, & \text{if } 300 \leq x \leq 400 \\
\frac{1}{80000} (2x - 700), & \text{if } 400 \leq x \leq 500 \\
\frac{1}{80000} (x^2 - 800x + 180000), & \text{if } 500 \leq x \leq 600 \\
\frac{1}{40000} (-x^2 + 1400x - 450000), & \text{if } 600 \leq x \leq 700 \\
1, & \text{if } x \geq 700.
\end{cases} \tag{25}
\]

Since $u_1(x) = -x^2 + 800x$, it follows from Eq. (12) and Theorem 2 that
\[
-(800 - H)^2 + 800(800 - H) = \int_{300}^{400} \left(-(800 - x)^2 + 800(800 - x)\right) d \left(\frac{1}{80000} (x - 300)^2\right) \\
+ \int_{400}^{500} \left(-(800 - x)^2 + 800(800 - x)\right) d \left(\frac{1}{800} (2x - 700)\right) \\
+ \int_{500}^{600} \left(-(800 - x)^2 + 800(800 - x)\right) d \left(\frac{1}{80000} (x^2 - 800x + 180000)\right) \\
\times \left(\frac{1}{40000} (-x^2 + 1400x - 450000)\right). \tag{26}
\]
Solving Eq. (26) yields $H^* \approx 555.46$ or $H^* \approx 245.54$.

On the other hand, from the insurer’s point of view, by Eq. (14) and Theorem 2
\[
1000 + 100 = \int_{300}^{400} (1000 + G - x + 100) d \left(\frac{1}{80000} (x - 300)^2\right) \\
+ \int_{400}^{500} (1000 + G - x + 100) d \left(\frac{1}{800} (2x - 700)\right) \\
+ \int_{500}^{600} (1000 + G - x + 100) d \left(\frac{1}{80000} (x^2 - 800x + 180000)\right) \\
\times \left(\frac{1}{40000} (-x^2 + 1400x - 450000)\right). \tag{27}
\]
Solving Eq. (27) yields $G_s = 525$. Since $H^* \geq G_s$ should hold, the feasible price area is $[525, 555.46]$.

4. Optimality of stop loss insurance under uncertain random environments

In all sorts of compensation rules, stop loss insurance (see Kaas et al., 2008, p. 8) is a kind of very simple but has practical value rule. The stop loss insurance is more effective than any other insurance with the same mean (same net premium) that operates on individual claims for risk averse insured, the result of which can refer to Kaas et al. (2008, p. 168) or Xie and Han (2000, p. 204).

In this section, we will prove that the stop loss insurance can also maximize the insured’s expected utility under uncertain random environments.

Assume $X$ is the potential loss of wealth, which is an uncertain random variable with chance distribution $\Phi(x)$. Let $I(X)$ be the claim in the insurance contract. According to Xie and Han (2000, p. 203), claim is the function of potential loss. We assume this rule still holds in uncertain random case, which implies that $I(X)$ is the function of $X$, $0 \leq I(X) \leq X$. Since $X$ is an uncertain random variable, $I(X)$ is also an uncertain random variable, then the net premium $E[I(X)]$ is defined as
\[
E[I(X)] = \int_0^{+\infty} I(x) d\Phi(x). \tag{28}
\]

Then we introduce the stop loss insurance under uncertain random loss $X$, in which a real number $d$ is fixed as the deductible, the claim denoted by $I_d(X)$, is defined as
\[
I_d(X) = \begin{cases} 
0, & \text{if } X \leq d \\
X - d, & \text{if } X > d.
\end{cases} \tag{29}
\]

From Eq. (28), the net premium of the stop loss insurance is
\[
E[I_d(X)] = \int_d^{+\infty} (x - d) d\Phi(x). \tag{30}
\]

**Theorem 3.** Assume the initial wealth $w$ of the insured is exposed to a risk $X$, which is an uncertain random variable with chance distribution $\Phi(x)$. It is also assumed that the insured is risk averse, which implies that his utility function $u(x)$ satisfies $u''(x) > 0$ and $u''(x) < 0$. Suppose that the insurance market provides all kinds of insurance contracts with claims $I(X)$, $0 \leq I(X) \leq X$, which are sold by net premiums. If the insured decides to buy an insurance contract with premium $P$, the optimal insurance is the stop loss contract $I_d(X)$ and the deductible $d^*$ is decided by
\[
P = \int_d^{+\infty} (x - d) d\Phi(x). \tag{31}
\]

**Proof.** When the claim is $I(X)$, the insured’s utility is denoted by $U(w - P - X + I(X))$, and when the claim is $I_d(X)$, the insured’s utility is denoted by $U(w - P - X + I_d(X))$ accordingly. The theorem is to prove that for any $I(X)$, $0 \leq I(X) \leq X$, satisfies $E[I(X)] = E[I_d(X)] = P$.

\[
U(w - P - X + I(X)) \leq U(w - P - X + I_d(X)) \tag{32}
\]
holds. Eq. (32) can be written as
\[
\int_0^{+\infty} u(w - P - x + I(x)) d\Phi(x) \leq \int_0^{+\infty} u(w - P - x + I_d(x)) d\Phi(x), \tag{33}
\]
i.e.,
\[
\int_0^{+\infty} [u(w - P - x + I(x)) - u(w - P - x + I_d(x))] d\Phi(x) \leq 0. \tag{34}
\]
Let $z = w - P - X + I(x)$ and $z^* = w - P - x + I_d(x)$. By Taylor expansion, we have
\[
u(z) = u(z^*) + u'(z^*)(z - z^*) + \frac{1}{2} u''(z^*)(z - z^*)^2 + o(|z - z^*|^2). \tag{35}
\]
Then
\[ u(z) - u(z^*) = u'(z^*)(z - z^*) + \frac{1}{2} u''(z^*)(z - z^*)^2 + o(|z - z^*|^2) \leq u'(z^*)(z - z^*) \quad \text{(since } u''(z^*) < 0) \]
\[ = u'(w - P + x + I_{E}(x)) \{ I(x) - I_{E}(x) \}. \quad (36) \]
If \( I(x) \leq I_{E}(x) \), the inequality (34) holds because of \( u'(x) > 0 \). If \( I(x) > I_{E}(x) \), from (29), we have
\[ X - I_{E}(x) = \begin{cases} X, & \text{if } X \leq d^* \\ d^*, & \text{if } X > d^*. \end{cases} \quad (37) \]
So \( w - P - x + I_{E}(x) \geq w - P - d^* \). Since \( u'(x) < 0 \), then
\[ u'(w - P - x + I_{E}(x)) \leq u'(w - P - d^*). \quad (38) \]
It follows from (36) and (38) that
\[ u(z) - u(z^*) \leq u'(w - P - d^*) \{ I(x) - I_{E}(x) \}. \quad (39) \]
Then we carry on integral on both side of inequality (39),
\[ \int_{0}^{\infty} [u(w - P + x + I_{E}(x)) - u(w - P - x + I_{E}(x))] d\Phi(x) \leq u'(w - P - d^*) \left[ \int_{0}^{\infty} I(x)d\Phi(x) - \int_{0}^{\infty} I_{E}(x)d\Phi(x) \right] \]
\[ = 0, \quad (40) \]
which implies that inequality (34) holds.

The proof is completed.

**Example 2.** Assume an insured has an initial wealth 1000 dollars, which is exposed to a potential loss \( X \). We also assume \( X = \eta + \tau \), in which direct loss \( \eta \) is uniformly distributed random variable \( U(200, 300) \) and indirect loss \( \tau \) is zigzag uncertain variable \( Z(100, 300, 400) \). So \( X \) is an uncertain random variable. The utility function of the insured is \( u(x) = \sqrt{x} \). Suppose that the premium the insured can pay is 240 dollars and there are all kinds of insurance contracts \( I(x) \), \( 0 \leq I(x) \leq X \) sold by net premiums. Then the optimal insurance policy will be decided.

Since \( u(x) = \sqrt{x} \), it is easy to see that \( u'(x) > 0 \) and \( u''(x) < 0 \). By Theorem 3, the insured should choose the stop loss contract
\[ I_{E}(x) = \begin{cases} 0, & \text{if } X \leq d^* \\ X - d^*, & \text{if } X > d^*. \end{cases} \quad (41) \]
and \( d^* \) is decided by
\[ 240 = \int_{d}^{700} (x - d) d\Phi(x), \quad (42) \]
in which \( \Phi(x) \) has been calculated in Example 1. When \( d \leq 300 \), we have
\[ 240 = \int_{d}^{400} (x - d) d\left( \frac{1}{80000}(x - 300)^2 \right) + \int_{400}^{500} (x - d) d\left( \frac{1}{800}(2x - 700) \right) + \int_{500}^{600} (x - d) d\left( \frac{1}{80000}(x^2 - 800x + 180000) \right) + \int_{600}^{700} (x - d) d\left( \frac{1}{40000}(-x^2 + 1400x - 450000) \right), \quad (43) \]
so \( d = 285 \). When \( 300 \leq d \leq 400 \), we have
\[ 240 = \int_{d}^{400} (x - d) d\left( \frac{1}{80000}(x - 300)^2 \right) + \int_{400}^{500} (x - d) d\left( \frac{1}{800}(2x - 700) \right) + \int_{500}^{600} (x - d) d\left( \frac{1}{80000}(x^2 - 800x + 180000) \right) + \int_{600}^{700} (x - d) d\left( \frac{1}{40000}(-x^2 + 1400x - 450000) \right), \quad (44) \]
so \( d = 1360.2 \) or \( d = 525 \) or \( d = -650 \), all the results are not in the given interval \([300, 400]\). When \( 400 \leq d \leq 500 \), we have
\[ 240 = \int_{d}^{500} (x - d) d\left( \frac{1}{800}(2x - 700) \right) + \int_{500}^{600} (x - d) d\left( \frac{1}{80000}(x^2 - 800x + 180000) \right) + \int_{600}^{700} (x - d) d\left( \frac{1}{40000}(-x^2 + 1400x - 450000) \right), \quad (45) \]
so \( d = 1686.6 \) or \( d = -186.6 \), all the results are not in the given interval \([400, 500]\). When \( 500 \leq d \leq 600 \), we have
\[ 240 = \int_{d}^{500} (x - d) d\left( \frac{1}{800}(2x - 700) \right) + \int_{500}^{600} (x - d) d\left( \frac{1}{80000}(x^2 - 800x + 180000) \right) + \int_{600}^{700} (x - d) d\left( \frac{1}{40000}(-x^2 + 1400x - 450000) \right), \quad (46) \]
so \( d = 885.3343 \) or \( d = 339.9197 \) or \( d = -25.2540 \), all the results are not in the given interval \([500, 600]\). When \( 600 \leq d \leq 700 \), we have
\[ 240 = \int_{d}^{700} (x - d) d\left( \frac{1}{80000}(x^2 - 800x + 180000) \right) + \int_{d}^{700} (x - d) d\left( \frac{1}{40000}(-x^2 + 1400x - 450000) \right), \quad (47) \]
“\( d^* \) does not exist any real solutions in this case.

In summary, only \( d = 285 \) is reasonable. So the optimal insurance is the stop loss contract with deductible \( d^* = 285 \).

**5. Conclusion**

In this paper, the potential loss of insured is assumed to be uncertain random variable and chance theory provides a mathematical foundation to deal with nondeterministic factors, which implies that we can use this method to solve more complex optimal insurance problems. Under the framework of chance theory, the upper and lower bounds on premium with uncertain random loss are given, respectively. Moreover, a mathematical model of uncertain random optimal insurance problem is established and the stop loss insurance is proved to be the optimal insurance policy, in which the deductible can be derived by solving the equation.

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**References**


