Adams–Simpson method for solving uncertain differential equation

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**A R T I C L E   I N F O**

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**A B S T R A C T**

Uncertain differential equation is a type of differential equation driven by canonical Liu process. How to obtain the analytic solution of uncertain differential equation has always been a thorny problem. In order to solve uncertain differential equation, early researchers have proposed two numerical algorithms based on Euler method and Runge–Kutta method. This paper will design another numerical algorithm for solving uncertain differential equations via Adams–Simpson method. Meanwhile, some numerical experiments are given to illustrate the efficiency of the proposed numerical algorithm. Furthermore, this paper gives how to calculate the expected value, the inverse uncertainty distributions of the extreme value and the integral of the solution of uncertain differential equation with the aid of Adams–Simpson method.

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**1. Introduction**

Probability theory, which was founded by Kolmogorov in 1933, has been widely used to model indeterminacy phenomena for a long time. A premise of applying probability theory is that the obtained probability distribution is close enough to the real frequency. However, due to some privacy or technological reasons, we usually have little or no sample data to estimate the probability distribution of a random variable. In this case, we have to invite some experts to give their belief degrees that each event will occur. A lot of surveys showed that human beings usually estimate a much wider range of values than the object actually takes (Liu [11]). This conservatism of human beings makes the belief degree deviate far from the frequency. Hence, it is inappropriate to treat the belief degree as a random variable and to model indeterminacy phenomena in this case by probability theory.


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As an important tool to deal with uncertain dynamic system, uncertain differential equation driven by canonical Liu process, was first proposed by Liu [7] in 2008. In recent years, many researchers have done a lot of work about uncertain differential equation. In 2010, Chen and Liu [1] proved a sufficient condition for the existence and uniqueness of the solution of uncertain differential equation. Then Gao [4] gave a weaker condition. In 2014, Gao and Yao [5] presented two continuity theorems on solution of uncertain differential equation. As far as we know, stability analysis of solutions is one of the central problems in uncertain differential equation. The concept of stability for uncertain differential equation was first given by Liu [8], and some stability theorems were proved by Yao et al. [19]. Later on, different types of stability of uncertain differential equation were explored, such as stability in mean (Yao et al. [22]) and stability in moment (Sheng-Wang [16]).

The solution methods of uncertain differential equation also draw much attention from a lot of researchers. Chen and Liu [1] provided the analytic solution of a linear uncertain differential equation in 2010. After that, Liu [12] and Yao [20] gave the analytic solutions for some special classes of nonlinear uncertain differential equation. In fact, it is difficult to obtain the analytic solution of general uncertain differential equation. Researchers turn to look for the numerical solution of uncertain differential equation. In 2013, Yao and Chen [18] found a way to transfer the known uncertain differential equation into a family of associated ordinary differential equations. What’s more, Yao and Chen [18] proposed famous Yao–Chen formula which can determine the inverse uncertainty distribution of the solution of an uncertain differential equation. Based on Yao–Chen formula, Yao [17] and Shen and Yao [14] presented Euler method and Runge–Kutta method to solve uncertain differential equation, respectively. Nowadays, uncertain differential equation has been applied to many areas especially in optimal control (Zhu [23] and Sheng et al. [15]) and finance (Liu [10] and Liu et al. [13]).

In this paper, we aim at providing another way to obtain the numerical solutions of uncertain differential equation via Adams–Simpson method. The remainder of this paper is organized as follows. The next section is intended to introduce some basic concepts and theorems in uncertainty theory and uncertain differential equation. Section 3 designs a numerical algorithm to solve uncertain differential equation. Based on Yao–Chen formula, the product uncertain measure was proved by Yao et al. [19]. Later on, different types of stability of uncertain differential equation were explored, such as stability in mean (Yao et al. [22]) and stability in moment (Sheng-Wang [16]).

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In this paper, we aim at providing another way to obtain the numerical solutions of uncertain differential equation via Adams–Simpson method. The remainder of this paper is organized as follows. The next section is intended to introduce some basic concepts and theorems in uncertainty theory and uncertain differential equation, which are used throughout this paper.

2. Preliminary

In this section, we introduce some basic concepts and theorems about uncertainty theory and uncertain differential equation, which are used throughout this paper.

2.1. Uncertainty theory

In order to provide a quantitative measurement that an uncertain phenomenon will occur, an axiomatic definition of uncertain measure is defined as follows.

**Definition 2.1** (Liu [6]). Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( \mathcal{M} \) is called an **uncertain measure** if it satisfies the following axioms:

- **Axiom 1. (Normality axiom)** \( \mathcal{M}(\Gamma) = 1 \);
- **Axiom 2. (Duality axiom)** \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any \( \Lambda \in \mathcal{L} \);
- **Axiom 3. (Subadditivity axiom)** For every countable sequence of \( \{\Lambda_i\} \in \mathcal{L} \), we have
  \[
  \mathcal{M}\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).
  \]

The triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an **uncertainty space**, and each element \( \Lambda \) in \( \mathcal{L} \) is called an **event**. Besides, the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) is defined by Liu [8] as follows:

1. **Axiom 4. (Product axiom)** Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying
  \[
  \mathcal{M}\left( \bigcap_{k=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Lambda_k)
  \]
  where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \ldots \), respectively.

In order to represent the quantities with uncertainty, an uncertain variable was proposed as a real valued function on an uncertainty space.

**Definition 2.2** (Liu [6]). An **uncertain variable** \( \xi \) is a measurable function from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set
\[
\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}
\]
is an event.
Definition 2.3 (Liu [6]). The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by
$$\Phi(x) = \mathcal{M}(\xi \leq x), \quad \forall x \in \mathbb{R}.$$ 

Definition 2.4 (Liu [9]). An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to $x$ at which $0 < \Phi(x) < 1$, and
$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.$$

In addition, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of $\xi$.

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. As one of the most important types of uncertain processes, the canonical Liu process is defined as follows.

Definition 2.5 (Liu [8]). An uncertain process $C_t$ ($t \geq 0$) is called a canonical Liu process if
1. $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
2. $C_t$ is a stationary independent increment process,
3. every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is
$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3t}}\right)\right)^{-1}.$$

Based on canonical Liu process, Liu integral is defined as an uncertain counterpart of Ito integral as follows.

Definition 2.6 (Liu [8]). Let $X_t$ be an uncertain process and let $C_t$ be a canonical Liu process. For any partition of the closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as
$$\Delta = \sup_{1 \leq i \leq k} |t_{i+1} - t_i|.$$ 

Then Liu integral of $X_t$ with respect to $C_t$ is defined by
$$\int_a^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be integrable.

2.2 Uncertain differential equation


Definition 2.7 (Liu [7]). Suppose $C_t$ is a canonical Liu process, and $f$ and $g$ are two functions. Given an initial value $X_0$,
$$dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t$$

is called an uncertain differential equation with an initial value $X_0$. The solution is an uncertain process $X_t$ that satisfies (1) identically in $t$.

Theorem 2.1 (Chen–Liu [1]). Let $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ be integrable uncertain processes. Then the linear uncertain differential equation
$$dX_t = (u_{1t}X_t + u_{2t}) \, dt + (v_{1t}X_t + v_{2t}) \, dC_t$$

has a solution
$$X_t = U_t \left( X_0 + \int_0^t \frac{u_{2s}}{U_s} \, ds + \int_0^t \frac{v_{2s}}{U_s} \, dC_s \right)$$

where
$$U_t = \exp \left( \int_0^t u_{1s} \, ds + \int_0^t v_{1s} \, dC_s \right).$$

Definition 2.8 (Yao–Chen [18]). The $\alpha$-path ($0 < \alpha < 1$) of an uncertain differential equation
$$dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t$$

with initial value $X_0$ is a deterministic function $X_t^\alpha$ with respect to $t$ that solves the corresponding ordinary differential equation
$$dX_t^\alpha = f(t, X_t^\alpha) \, dt + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) \, dt$$

(2)
where
\[ \Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \]

**Theorem 2.2** (Yao–Chen formula [18]). Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of uncertain differential Eq. (1), respectively. Then
\[
\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha,
\]
\[
\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.
\]
In addition, Yao and Chen found that the inverse uncertainty distribution \( \Psi_t^{-1}(\alpha) \) of the solution \( X_t \) is just its \( \alpha \)-path \( X_t^\alpha \).

On the basis of Yao–Chen formula, some formulas to calculate the expected value, the inverse uncertainty distributions of the extreme value and the integral of solution of an uncertain differential equation are presented by following theorems.

**Theorem 2.3** (Yao-Chen [18]). Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of uncertain differential Eq. (1), respectively. Then for any monotone (increasing or decreasing) function \( J(x) \), we have
\[
E[J(X_t)] = \int_0^1 J(X_t^\alpha) d\alpha.
\]

**Theorem 2.4** (Yao [17]). Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of uncertain differential Eq. (1), respectively. Then for any time \( T > 0 \) and a strictly increasing (decreasing) function \( J(x) \), the supremum
\[
\sup_{0 \leq t \leq T} J(X_t)
\]
has an inverse uncertainty distribution
\[
\Psi_T^{-1}(\alpha) = \sup_{0 \leq t \leq T} J(X_t^\alpha) \left( \Psi_T^{-1}(\alpha) = \sup_{0 \leq t \leq T} J(X_t^{1-\alpha}) \right),
\]
and the infimum
\[
\inf_{0 \leq t \leq T} J(X_t)
\]
has an inverse uncertainty distribution
\[
\Phi_T^{-1}(\alpha) = \inf_{0 \leq t \leq T} J(X_t^\alpha) \left( \Phi_T^{-1}(\alpha) = \inf_{0 \leq t \leq T} J(X_t^{1-\alpha}) \right).
\]

**Theorem 2.5** (Yao [17]). Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of uncertain differential Eq. (1), respectively. Then for any time \( T > 0 \) and a strictly increasing (decreasing) function \( J(x) \), the integral
\[
\int_0^T J(X_t) dt
\]
has an inverse uncertainty distribution
\[
\Psi_T^{-1}(\alpha) = \int_0^T J(X_t^\alpha) dt \left( \Psi_T^{-1}(\alpha) = \int_0^T J(X_t^{1-\alpha}) dt \right).
\]

### 3. Adams–Simpson method for uncertain differential equation

By Theorem 2.2, we know that uncertain differential Eq. (1) can be transferred into a family of associated ordinary differential equations, and Yao–Chen formula can determine the inverse uncertainty distribution of the solution of (1). Furthermore, when an ordinary differential equation has no analytical solution, we can give its numerical solution by linear multistep method. In this section, we first introduce double-step and second-order explicit Adams formula and double-step and fourth-order implicit Simpson formula for ordinary differential equation. Based on Yao–Chen formula, we then link Adams formula and Simpson formula and design Adams–Simpson method to solve the uncertain differential equation without analytical solution in Section 3.1. To illustrate the efficiency of Adams–Simpson method for solving the uncertain differential equation, we give an experiment in Section 3.2.

#### 3.1. Adams–Simpson method

For an ordinary differential equation with initial value \( X_0 \)
\[
dX = F(t, X) dt
\]
some notations and parameters used in the later parts are introduced as follows,
\[ t_i; t_i = t_0 + ih, \text{ where } h \text{ is a fixed step length, } i = 0, 1, 2, \ldots; \]
\(X_i\): an approximation to \(X\) at \(t_i\);
\(\hat{X}_i\): the predicted value of \(X_i\);
\(\tilde{X}_i\): the corrected value of \(X_i\);
\(F_i^e: F(t_i, X_i)\);
\(F_i^c: F(t_i, \hat{X}_i)\);
\(\Phi_i^{−1}(\alpha) = \sqrt{3} \ln \frac{\alpha}{1−\alpha}\).

The double-step and second-order explicit Adams formula can be written as

\[
X_{i+1} = X_i + \frac{h}{2} (3F_i^e - F_{i-1}) \tag{4}
\]

The double-step and fourth-order implicit Simpson formula is

\[
X_{i+1} = X_{i-1} + \frac{h}{3} (F_{i+1} + 4F_i^e + F_{i-1}) \tag{5}
\]

Making a connection between formula (4) and (5), we obtain a predictor–corrector formula

\[
\begin{align*}
\hat{X}_{i+1} &= X_i + \frac{h}{2} (3F_i^e - F_{i-1}) \\
\tilde{X}_{i+1} &= X_{i-1} + \frac{h}{3} (F_{i+1} + 4F_i^e + F_{i-1})
\end{align*} \tag{6}
\]

Here, we call formula (6) as Adams–Simpson formula.

Now, consider the uncertain differential equation

\[
dX_t = f(t, X_t) dt + g(t, X_t) dC_t
\]

with initial value \(X_0\). Suppose that the coefficients \(f(t, x)\) and \(g(t, x)\) satisfy the linear growth condition and Lipschitz condition. That is, the uncertain differential Eq. (1) has a unique solution, and its \(\alpha\)-path is

\[
\begin{align*}
\hat{X}^{\alpha}_t &= f(t, X^{\alpha}_t) dt + |g(t, X^{\alpha}_t)| \Phi_i^{−1}(\alpha) dt \\
\hat{X}^{\alpha}_0 &= X_0.
\end{align*} \tag{7}
\]

For the sake of simplicity, we write

\[
F(t, X^{\alpha}_t) = f(t, X^{\alpha}_t) + |g(t, X^{\alpha}_t)| \Phi_i^{−1}(\alpha).
\]

Then the ordinary differential Eq. (7) is reduced to

\[
\begin{align*}
\hat{X}^{\alpha}_t &= F(t, X^{\alpha}_t) dt \\
\hat{X}^{\alpha}_0 &= X_0.
\end{align*}
\]

On the basis of Theorem 2.2 and Adams–Simpson formula, we will present Adams–Simpson method for solving uncertain differential Eq. (1) as follows.

**Initialization:**
Fix a time \(T\), a number of iteration \(N\) and a step length \(h = T/N\). Set the subscript variable \(i = 1\), \(\alpha = 0\), \(\Delta \alpha = 0.01\). \(X^{\alpha}_0 = X_0\) and \(X^{\alpha}_1 = X^{\alpha}_0 + hF_0\).

**Iteration:**
Repeat
Repeat
Calculate

\[
\begin{align*}
\hat{X}^{\alpha}_{i+1} &= X^{\alpha}_i + \frac{h}{2} (3F_i^e - F_{i-1}) \\
\tilde{X}^{\alpha}_{i+1} &= X^{\alpha}_{i-1} + \frac{h}{3} (F_{i+1} + 4F_i^e + F_{i-1})
\end{align*}
\]

and let \(i = i + 1\);

**Until** \(i = N\), and obtain \(X^{\alpha}_N = X^{\alpha}_N\).

Let \(\alpha = \alpha + \Delta \alpha\). \(X^{\alpha}_0 = X_0\) and \(X^{\alpha}_1 = X^{\alpha}_0 + hF_0\);

**Until** \(\alpha\) reaches 1.

Finally, we get a 99-table at a fixed time \(T\):
Fig. 1. Comparison of uncertainty distribution between the analytic solution (9) and three numerical solutions at \( T = 1 \), where (a), (b), and (c) show the uncertainty distribution between the analytic solution (9) and the numerical solution produced by Euler method, Runge–Kutta method and Adams–Simpson method, respectively.

The above table gives an approximate inverse uncertainty distribution of \( X_T \), that is, we can find \( \alpha \)-path \( X^\alpha_T \) from the table such that \( \mathcal{M}(X_T \leq X^\alpha_T) = \alpha \) for any \( \alpha = i/100 \) (\( i = 1, 2, \ldots, 99 \)). If \( \alpha \neq i/100 \), then it is suggested to employ a numerical interpolation method to get an approximate \( X^\alpha_T \) (\( i = 1, 2, \ldots, 99 \)). When \( \Delta \alpha \) is taken as 0.001 or 0.0001, the 99-table can be extended to 999-table or 9999-table and a more precise inverse uncertainty distribution of \( X_T \) will be produced.

3.2. Numerical experiments

In this section, we will compare the numerical results from Euler, Runge–Kutta and Adams–Simpson methods with the analytic result by solving the same uncertain differential equation which can be solved explicitly, and analyze the relationship between some parameters and the numerical solution produced by Adams–Simpson method.

**Example 3.1.** Consider a linear uncertain differential equation
\[
dX_t = X_t \, dt + X_t \, dC_t
\]
with initial value \( X_0 = 1 \).

According to Theorem 2.1, we can know that the analytic solution of (8) is
\[
X_t = X_0 \exp(t + C_t).
\]

The inverse uncertainty distribution of \( X_t \) is
\[
\Psi_t^{-1}(\alpha) = \exp(t + \Phi^{-1}(\alpha)t),
\]
that is just the \( \alpha \)-path \( X^\alpha_t \) of the uncertain differential Eq. (8).

Next we will compare the analytic solution (9) with three numerical solutions produced by Euler, Runge–Kutta and Adams–Simpson methods at a fixed time \( T = 1 \) (see Fig. 1).

From Fig. 1, we can see that the three uncertainty distributions produced respectively by Euler, Runge–Kutta and Adams–Simpson methods are very close to the uncertainty distribution of the analytic solution (9). Hence, Euler, Runge–Kutta and Adams–Simpson methods can solve the uncertain differential equation. In addition, we further observe Fig. 1, and find that the uncertainty distribution produced by Adams–Simpson method is closer to the analytical solution compared to the other two uncertainty distributions produced by Euler method and Runge–Kutta method, respectively. So it is clear that Adams–Simpson method has more higher accuracy than Euler and Runge–Kutta methods.

From the above analysis, we can see that Adams–Simpson method is a better way to solve the uncertain differential equation than Euler method and Runge–Kutta method. Since there are two parameters \((h, \Delta \alpha)\) in Adams–Simpson method, we then begin to study the relationship between parameters \((h, \Delta \alpha)\) and the solution of uncertain differential Eq. (8) at a fixed time \( T = 1 \). In the following, we first consider one case where \( h \) is fixed to be 0.01 and \( \Delta \alpha \) is changing, the corresponding solutions produced by Adams–Simpson method are shown in Fig. 2. Another case, where \( \Delta \alpha \) is fixed to be 0.01 and \( h \) is changing, is then considered and the corresponding solutions produced by Adams–Simpson method are shown in Fig. 3.

Fig. 2 shows that a more accurate uncertainty distribution can be obtained when the parameter \( \Delta \alpha \) takes smaller values, while Fig. 3 tells us that we can improve the accuracy of the numerical solution if we use a smaller step length \( h \).
4. Applications

In this section, we apply Adams–Simpson method for calculating the expected value, the inverse uncertainty distributions of the extreme value and the integral of the special uncertain process \( X_t \) which is the solution of uncertain differential Eq. (1).

4.1. Adams–Simpson method for expected value

According to Theorem 2.3, when \( f(x) \) is a monotone (increasing or decreasing) function, we design a numerical algorithm for calculating the expected value of \( f(X_t) \) via Adams–Simpson method as follows.

Initialization:

Fix a time \( t \), a number of iteration \( N \) and a step length \( h = \frac{t}{N} \). Set the subscript variable \( i = 1 \), \( \alpha = 0 \), \( \Delta \alpha = 0.01 \), \( X_\alpha^0 = X_0 \) and \( X_\alpha^1 = X_\alpha^0 + hF_0 \).

Iteration:

Repeat

Repeat

Calculate

\[
\begin{align*}
X_{\alpha i+1}^\alpha &= X_{\alpha i}^\alpha + \frac{h}{2} (3F_i - F_{i-1}) \\
X_{\alpha i+1}^{\alpha -1} &= X_{\alpha i-1}^{\alpha -1} + \frac{h}{2} (F_{i+1} + 4F_i + F_{i-1})
\end{align*}
\]

and let \( i = i + 1 \);

Until \( i = N \), and obtain \( X_{\alpha N}^\alpha = X_{\alpha N}^{\alpha -1} \);

Let \( \alpha = \alpha + \Delta \alpha \), \( X_0^\alpha = X_0 \) and \( X_1^\alpha = X_0^\alpha + hF_0 \);

Until \( \alpha \) reaches 1.
Finally, the expected value of $J(X_t)$ can be determined by

$$E[J(X_t)] = \sum_{\alpha_j} J(X_\alpha t) \cdot \Delta \alpha,$$

where $\alpha_j = \Delta \alpha \cdot j$ ($j = 0, 1, \ldots, 100$).

**Example 4.1.** Consider the following uncertain differential equation

$$dX_t = X_t dt + X_t dC_t$$

with initial value $X_0 = 1$.

Let $J(x) = 4\sqrt{x}$. By Adams–Simpson method, the expected value of $J(X_t)$ is shown in Fig 4, where $t = 1, 2, \ldots, 10$.

### 4.2. Adams–Simpson method for extreme value

According to Theorem 2.4, when $J(x)$ is a strictly increasing function, we design a numerical algorithm for calculating the inverse uncertainty distribution of the supremum

$$\text{sup}_{0 \leq t \leq T} J(X_t)$$

and the infimum

$$\text{inf}_{0 \leq t \leq T} J(X_t)$$

via Adams–Simpson method as follows.

**Initialization:**

Fix a time $T$, $\alpha$ in $[0, 1]$, a number of iteration $N$ and a step length $h = T/N$. Set the subscript variable $i = 1$, $X_0^\alpha = X_0$, $X_1^\alpha = X_0^\alpha + hF_0$, $H_{\text{sup}} = \max\{J(X_0), J(X_1^\alpha)^\alpha\}$ and $H_{\text{inf}} = \min\{J(X_0), J(X_1^\alpha)\}$.

**Iteration:**

Repeat

Calculate

$$\begin{cases}
X_{i+1}^\alpha = X_i^\alpha + \frac{h}{2} (3F_i - F_{i-1}) \\
X_{i+1}^\alpha = X_i^\alpha + \frac{h}{2} (F_i + 4F_i + F_{i-1})
\end{cases}$$

Let $X_{i+1}^\alpha = X_{i+1}^\alpha$. $H_{\text{sup}} = \max\{H_{\text{sup}}, J(X_{i+1}^\alpha)\}$, $H_{\text{inf}} = \min\{H_{\text{inf}}, J(X_{i+1}^\alpha)\}$ and $i = i + 1$;

Until $i = N$.

Thus, the inverse uncertainty distributions of $\text{sup}_{0 \leq t \leq T} J(X_t)$ and $\text{inf}_{0 \leq t \leq T} J(X_t)$ are $\Psi_T^{-1}(\alpha) = H_{\text{sup}}$ and $\Upsilon_T^{-1}(\alpha) = H_{\text{inf}}$, respectively.

Similarly, when $J(x)$ is a strictly decreasing function, we can also design a numerical method for calculating the inverse uncertainty distribution of the supremum

$$\text{sup}_{0 \leq t \leq T} J(X_t)$$
Fig. 5. The uncertainty distribution of \( \sup_{0 \leq t \leq T} J(X_t) \) with \( T = 10 \).

and the infimum
\[
\inf_{0 \leq t \leq T} J(X_t).
\]

**Initialization:**
Fix a time \( T, \alpha \) in \([0, 1]\), a number of iteration \( N \) and a step length \( h = T/N \). Set the subscript variable \( i = 1, X_0^{1-\alpha} = X_0, X_1^{1-\alpha} = X_0^{1-\alpha} + hF_0 \), \( H_{\sup} = \max\{J(X_0), J(X_1^{1-\alpha})\} \) and \( H_{\inf} = \min\{J(X_0), J(X_1^{1-\alpha})\} \).

**Iteration:**

Repeat

Calculate
\[
\begin{align*}
X_{i+1}^{1-\alpha} &= X_i^{1-\alpha} + \frac{h}{2}(3F_i - F_{i-1}) \\
X_{i+1}^{1-\alpha} &= X_i^{1-\alpha} + \frac{h}{2}(F_{i+1} + 4F_i + F_{i-1})
\end{align*}
\]

Let \( X_{i+1}^{1-\alpha} = X_{i+1}^{1-\alpha} \), \( H_{\sup} = \max\{H_{\sup}, J(X_{i+1}^{1-\alpha})\} \), \( H_{\inf} = \min\{H_{\inf}, J(X_{i+1}^{1-\alpha})\} \) and \( i = i + 1 \);

Until \( i = N \).

Finally, the inverse uncertainty distributions of \( \sup_{0 \leq t \leq T} J(X_t) \) and \( \inf_{0 \leq t \leq T} J(X_t) \) can be determined by \( \Psi_T^{-1}(\alpha) = H_{\sup} \) and \( T_T^{-1}(\alpha) = H_{\inf} \), respectively.

**Example 4.2.** Consider the following uncertain differential equation
\[
dX_t = X_t dt + X_t dC_t
\]
with initial value \( X_0 = 1 \).

Let \( J(x) = \sqrt{x} \). By Adams–Simpson method, the uncertainty distribution of \( \sup_{0 \leq t \leq T} J(X_t) \) with \( T = 10 \) is shown in Fig. 5.

### 4.3. Adams–Simpson method for integral

Based on **Theorem 2.5**, when \( J(x) \) is a strictly increasing function, we design a numerical algorithm for calculating the inverse uncertainty distribution of the integral
\[
\int_0^T J(X_t) dt
\]
via Adams–Simpson method as follows.

**Initialization:**
Fix a time \( T, \alpha \) in \([0, 1]\), a number of iteration \( N \) and a step length \( h = T/N \). Set the subscript variable \( i = 1, X_0^{\alpha} = X_0 \) and \( X_i^{\alpha} = X_0^{\alpha} + hF_0 \).
Iteration:
Repeat
Calculate
\[
\begin{aligned}
X_{i+1}^{\alpha} &= X_i^{\alpha} + \frac{h}{2} (3F_i - F_{i-1}) \\
X_{i+1}^{\alpha} &= X_{i-1}^{\alpha} + \frac{h}{2} (F_{i+1} + 4F_i + F_{i-1})
\end{aligned}
\]
Obtain \( X_{i+1}^{\alpha} = X_{i+1}^{\alpha} \cdot J(X_{i+1}^{\alpha}) \), and let \( i = i + 1 \);
Until \( i = N \).
Thus, the time integral
\[
\int_0^T J(X) \, dt
\]
has the inverse uncertainty distribution
\[
\Psi^{-1}_T(\alpha) = \sum_{i=1}^{N} J(X_{i+1}^{\alpha})h.
\]
Similarly, for a strictly decreasing function \( J(x) \), we can also design a numerical method for calculating the inverse uncertainty distribution of the integral
\[
\int_0^T J(X) \, dt
\]
via Adams–Simpson method as follows.
Initialization:
Fix a time \( T, \alpha \) in \([0, 1]\), a number of iteration \( N \) and a step length \( h = T/N \). Set the subscript variable \( i = 1 \), \( X_0^{1-\alpha} = X_0 \) and \( X_1^{1-\alpha} = X_0^{1-\alpha} + hF_0 \).
Iteration:
Repeat
Calculate
\[
\begin{aligned}
X_{i+1}^{1-\alpha} &= X_i^{1-\alpha} + \frac{h}{2} (3F_i - F_{i-1}) \\
X_{i+1}^{1-\alpha} &= X_{i-1}^{1-\alpha} + \frac{h}{2} (F_{i+1} + 4F_i + F_{i-1})
\end{aligned}
\]
Obtain \( X_{i+1}^{1-\alpha} = X_{i+1}^{1-\alpha} \cdot J(X_{i+1}^{1-\alpha}) \), and let \( i = i + 1 \);
Until \( i = N \).
Finally, the inverse uncertainty distribution of the time integral
\[
\int_0^T J(X) \, dt
\]
is determined by
\[ \Psi_T^{-1}(\alpha) = \sum_{i=1}^{N} J(X_i^{1-\alpha})h. \]

**Example 4.3.** Consider the following uncertain differential equation
\[ dX_t = X_t dt + X_t dC_t \]
with initial value \( X_0 = 1 \).

Let \( J(x) = \sqrt{x} \). By Adams–Simpson method, the uncertainty distribution of the integral \( \int_0^T J(X_t)dt \) with \( T = 5 \) is shown in Fig. 6.

5. Conclusions

In this paper, we proposed a new numerical algorithm to solve uncertain differential equation by Adams–Simpson method. After that, one numerical example was given to illustrate that our proposed method is more accurate than Euler and Runge–Kutta methods. Furthermore, we designed some algorithms to calculate the expected value, the inverse uncertainty distribution of the extreme value and the integral of the special uncertain process which is the solution of a general uncertain differential equation.

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**References**