Nested Uncertain Differential Equations and Its Application to Multi-factor Term Structure Model

Xiaowei Chen

International Business School, Nankai University, Tianjin 30071, China

School of Finance, Nankai University, Tianjin 30071, China

chenx@nankai.edu.cn

Abstract

Uncertain differential equation is a type of differential equation driven by Liu process. Compared to one dimensional uncertain differential equations, more situations can be considered by multi-dimension uncertain differential equations to accurately describe multiple risk processes. In this manuscript, a special type of high dimensional uncertain differential equations, named nested uncertain differential equations, are introduced. The α-path of the proposed nested uncertain differential equation will be derived. Furthermore, a numerical method is designed for solving nested uncertain differential equations using the proposed α-path method. We also used nested uncertain differential equations to build a multi-factor term structure model for short interest rate where the drift is described by another uncertain differential equation. The pricing of the zero-coupon bond will calculated based on the proposed model.

Keywords: Uncertainty theory; numerical method; nested uncertain differential equations; multi-factor term structure.

1 Introduction

Stochastic differential equation is a type of differential equation driven by Wiener process. It has a wide applications including economy, biology, physics and automation fields. As we known, it is very difficult to find the analytical solution for most of the stochastic differential equations. Numerical solution becomes an alternative. The proposed Euler-Maruyama approximation (Maruyama [11]), Milstein approximation (Milstein [12]) and Runge-Kutta approximation (Rumelin [14]) were designed based on different types of recursion formulas. Euler-Maruyama approximation uses first-order Taylor expansion as the recursion formula, Milstein approximation uses second-order Taylor expansion, and Runge-Kutta approximation uses Runge-Kutta expansion. The numerical algorithms make the application of stochastic differential equations become more convenient.

Stochastic process, stochastic calculus and stochastic differential equation are all based on the probability measure with the additivity property. The probability distribution is estimated by historical data. When no samples are available, some domain experts to are invited to evaluate the belief degree that each event will happen. In order to rationally deal with personal degree of belief, uncertainty theory was founded in 2007 by Liu [3] based on normality, duality, subadditivity and product axioms. As an uncertain counterpart of Wiener process, Liu [5] introduced canonical Liu process that is a Lipschitz continuous uncertain process with stationary and independent increments where the increments follow normal uncertainty distribution rather than Gaussian distribution. To deal with the differentiation and integration of functions of uncertain processes, Liu [5] proposed uncertain calculus in 2009. In 2012, Yao [15] proposed uncertain calculus with respect to renewal process. Chen [2] studied uncertain calculus with respect to uncertain independent and stationary increments processes.

Uncertain differential equation is a type of differential equation driven by canonical Liu process. It was proposed by Liu [4] in 2008. The existence and uniqueness of solution of uncertain differential equations were first proved by Chen and Liu [1], and stability analysis of uncertain differential equations was given by Yao *et. al* [17]. Multi-dimension uncertain differential equation was introduced by Yao [18]. Uncertain differential equation was first applied to finance by Liu [5] to model stock price. Peng and Yao [13] proposed another stock model to describe the stock price in long-run, and Liu *et. al.* [10] proposed a currency model in uncertain environment. Besides, Zhu [19] developed uncertain optimal control theory via uncertain differential equations. The numerical methods for uncertain differential equations were built by Yao and Chen [16] using the α-path methods.

The analytic solution methods and numerical methods for multi-dimension uncertain differential equation is more difficult to build. We will study a special type of multi-dimension uncertain differential equation in this paper. The numerical methods to solve proposed nested uncertain differential equations will be designed. We will also employ the nested uncertain differential equations to model the multi-factor term structure model. The remainder of this paper is organized as follows. Some basic concepts and properties are reviewed in Section 2. Section 3 introduces nested uncertain differential equation and derives its α-path, and Section 4 designs a 99-method to solve nested uncertain differential equations. A multi-factor term structure model is built in Section 5. A brief summary is included in Section 6.

2 Preliminary

The uncertainty theory is branch of mathematics based on normality, duality and subadditivity axioms.

Definition 1. *(Liu [3]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ was defined by Liu [5] as follows,
Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Then the product

uncertain measure $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{i=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k=1,2,\cdots$, *respectively.*

The triplet $(\Gamma,\mathcal{L},\mathcal{M})$ is called an uncertainty space. Based on the axioms of uncertain measure, an uncertainty theory was founded by Liu [3] in 2007 and refined by Liu [7] in 2010. An uncertain variable is a function from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers. The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x)=\mathcal{M}\{\xi\le x\}$$

for any real number x. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0<\Phi(x)<1$, and

$$\lim_{x\to-\infty}\Phi(x)=0,\ \lim_{x\to+\infty}\Phi(x)=1.$$

Definition 2. *(Liu [7]) Let* ξ *be an uncertain variable with regular uncertainty distribution* $\Phi(x)$. *Then the inverse function* $\Phi^{-1}(\alpha)$ *is called the inverse uncertainty distribution of* ξ.

Let ξ be an uncertain variable with an uncertainty distribution Φ. If the expected value exists, it was proved by Liu [7] that

$$E[\xi]=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

An uncertain process is essentially a sequence of uncertain variables indexed by time. The study of uncertain process was started by Liu [4] in 2008.

Definition 3. *(Liu [4]) Let* T *be an index set and let* $(\Gamma,\mathcal{L},\mathcal{M})$ *be an uncertainty space. An uncertain process is a measurable function from* $T\times(\Gamma,\mathcal{L},\mathcal{M})$ *to the set of real numbers, i.e., for each* $t\in T$ *and any Borel set* B *of real numbers, the set*

$$\{X_t\in B\}=\{\gamma\in\Gamma|X_t(\gamma)\in B\}$$

is an event.

Definition 4. *(Liu [5]) An uncertain process* C_t $(t\ge 0)$ *is said to be a canonical process if*
(i) $C_0=0$ *and almost all sample paths are Lipschitz continuous,*
(ii) C_t *has stationary and independent increments,*
(iii) every increment $C_{s+t}-C_s$ *is a normal uncertain variable with expected value 0 and variance* t^2, *whose uncertainty distribution is*

$$\Phi(x)=\left(1+\exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1},\ x\in\Re.$$

Based on canonical process, uncertain integral and uncertain differential were then defined by Liu [5], thus offering a theory of uncertain calculus.

Definition 5. *(Liu [5]) Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Definition 6. *(Liu [5]) Let C_t be a canonical process and Z_t be an uncertain process. If there exist uncertain processes μ_s and σ_s such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$$

for any $t \ge 0$, then Z_t is said to be differentiable and have an uncertain differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

Liu [5] verified the fundamental theorem of uncertain calculus, i.e., for a canonical process C_t and a continuous differentiable function $h(t, c)$, the uncertain process $Z_t = h(t, C_t)$ has an uncertain differential

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

Based on the fundamental theorem, Liu proved the integration by parts theorem, i.e., for two differentiable uncertain process X_t and Y_t, the uncertain process $X_t Y_t$ has an uncertain differential

$$\mathrm{d}(X_t Y_t) = Y_t \mathrm{d}X_t + X_t \mathrm{d}Y_t.$$

Uncertain calculus provides a theoretical foundation for constructing uncertain differential equations. Instead of using Wiener process, a differential equation driven by canonical process is defined as follows.

Definition 7. *(Liu [4]) Suppose C_t is a canonical process, and f and g are some given functions. Given an initial value X_0, then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation with an initial value X_0. A solution is an uncertain process X_t that satisfies Eq. (1) identically in t.

Theorem 1. *(Existence and Uniqueness Theorem, Chen and Liu [1]) The uncertain differential equation (1) has a unique solution if the coefficients $f(x, t)$ and $g(x, t)$ satisfy the Lipschitz condition*

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \le L|x - y|$$

for all $x, y \in \Re, t \ge 0$ and linear growth condition

$$|f(x, t)| + |g(x, t)| \le L(1 + |x|)$$

for all $x \in \Re, t \ge 0$ for some constants L. Moreover, the solution is sample-continuous.

It has been proved by Chen and Liu [1] that the linear uncertain differential equation

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t$$

has a solution

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right)$$

where

$$U_t = \exp\left(\int_0^t u_{1r}\mathrm{d}r + \int_0^t v_{1r}\mathrm{d}C_r\right).$$

Definition 8. *(Yao [18]) Let C_t be an n-dimensional canonical Liu process. Suppose $f(t,\boldsymbol{x})$ is a vector-valued function from $T \times \Re^m$ to $\Re^m$, and $g(t,\boldsymbol{x})$ is a matrix-valued function from $T \times \Re^m$ to the set of $m \times n$ matrices. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{2}$$

is called an m-dimensional uncertain differential equation driven by an n-dimensional canonical Liu process. A solution is an m-dimensional uncertain process that satisfied (2) identically in each t.

The concept of α-path is introduced as follows.

Definition 9. *(Yao and Chen [16])The α-path $(0 < \alpha < 1)$ of an uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

with initial value X_0 is a deterministic function X_t^α with respect to t that solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1.$$

Theorem 2. *(Yao and Chen [16]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

respectively. Then

$$\mathcal{M}\{X_s \le X_s^\alpha,\ \forall t > 0\} = \alpha, \tag{3}$$

$$\mathcal{M}\{X_s > X_s^\alpha,\ \forall t > 0\} = 1 - \alpha. \tag{4}$$

The proved formulas show that the solution of an uncertain differential equation is related to a class of ordinary differential equations.

3 Nested Uncertain Differential Equations

As we known, it is very difficult to obtain analytical solutions for most of the multiple dimensional uncertain differential equations. Similarly, it is also very difficult to obtain numerical solutions. Now, we will study a special kind of two dimensional uncertain differential equations called nested uncertain differential equation. Suppose that Y_t is the solution of uncertain differential equation

$$\mathrm{d}Y_t = f_1(t, Y_t)\mathrm{d}t + g_1(t, Y_t)\mathrm{d}C_{1t} \tag{5}$$

where $f_1(t, x)$ and $g_1(t, x)$ are two given functions and C_{1t} is a canonical Liu process.

Definition 10. *Assume that $f(t, x, y)$ and $g(t, x, y)$ are two given functions, C_{2t} is a canonical Liu processes independent with C_{1t}. The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t, Y_t)\mathrm{d}t + g(t, X_t, Y_t)\mathrm{d}C_{2t} \tag{6}$$

is called a nested uncertain differential equation where Y_t is the solution to equation (5).

In order to study the α-path of the proposed nested uncertain differential equation (6), the useful lemma is first introduced here.

Lemma 1. *Assume that $f(t, x, y)$ and $g(t, x, z)$ are continuous functions. Let $\phi(t)$ be a solution of the ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, y) + K|g(t, x)|, \quad x(0) = x_0$$

where K is a real number. Let $\psi(t)$ be a solution of the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, y) + k(t)g(t, x), \quad x(0) = x_0$$

where $k(t)$ is a real function.
i) If $k(t)g(t, x, z) \leq K|g(t, x)|$ for $t \in [0, T]$, then $\psi(T) \leq \phi(T)$.
ii) If $k(t)g(t, x, z) > K|g(t, x)|$ for $t \in [0, T]$, then $\psi(T) > \phi(T)$.

Let Y_t^α be α-path of the uncertain differential equations

$$\mathrm{d}Y_t = f_1(t, Y_t)\mathrm{d}t + g_1(t, Y_t)\mathrm{d}C_{1t}.$$

Theorem 3. *Assume that $f(t, x, y)$ is a continuous function monotone increasing with respect to y and $g(t, x)$ is a continuous function monotone. Then the α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t, Y_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is X_t^α subject to

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha, Y_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t.$$

Then

$$\mathcal{M}\{X_t \leq X_t^\alpha,\ \forall t > 0\} = \alpha, \tag{7}$$

$$\mathcal{M}\{X_t > X_t^\alpha,\ \forall t > 0\} = 1 - \alpha. \tag{8}$$

Proof: Write

$$A=\{t\in[0,s]|g\left(t,X_t^\alpha\right)\geq 0\},$$

$$B=\{t\in[0,s]|g\left(t,X_t^\alpha\right)<0\}.$$

Then it is obvious that $A\cup B=[0,s]$. Write

$$\Lambda_1=\left\{\gamma_2\left|\frac{\mathrm{d}C_t(\gamma_2)}{\mathrm{d}t}\leq\Phi^{-1}(\alpha)\text{ for }t\in(0,u]\right.\right\},$$

$$\Lambda_2=\left\{\gamma_2\left|\frac{\mathrm{d}C_t(\gamma_2)}{\mathrm{d}t}\geq\Phi^{-1}(1-\alpha)\text{ for }t\in(0,u]\right.\right\}$$

where Φ^{-1} is the inverse uncertainty distribution of $\mathcal{N}(0,1)$. Since C_t is an independent increment process, we get $\mathcal{M}\{\Lambda_1\cap\Lambda_2\}=\alpha$. For any $\gamma_1\in\{Y_t(\gamma_1)\leq Y_t^\alpha\}$ and $\gamma_2\in\Lambda_1\cap\Lambda_2$, we have

$$f(t,X_t(\gamma_2),Y_t(\gamma_1))+g(t,X_t(\gamma_2))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t}\leq f(t,X_t^\alpha,Y_t^\alpha)+|g(t,X_t^\alpha)|\Phi^{-1}(\alpha),t\in[0,s].$$

It follows from Lemma 1 that $X_t(\gamma)\leq X_t^\alpha$, so $\{Y_t(\gamma_1)\leq Y_t\}\cap(\Lambda_1\cap\Lambda_2)\subset\{X_t\leq X_t^\alpha\}$, and

$$\mathcal{M}\{X_s\leq X_s^\alpha\}\geq\mathcal{M}\{\{Y_t\leq Y_t^\alpha\}\cap(\Lambda_1\cap\Lambda_2)\}=\alpha. \tag{9}$$

Write

$$\Lambda_3=\left\{\gamma\left|\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t}>\Phi^{-1}(\alpha)\text{ for }t\in(0,u]\right.\right\},$$

$$\Lambda_4=\left\{\gamma\left|\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t}<\Phi^{-1}(1-\alpha)\text{ for }t\in(0,u]\right.\right\}.$$

Then we get $\mathcal{M}\{\Lambda_3\cap\Lambda_4\}=1-\alpha$. For any $\gamma_1\in\{Y_t(\gamma_1)<Y_t^{1-\alpha}\}$ and $\gamma_2\in\Lambda_3\cap\Lambda_4$, we have

$$f(t,X_t(\gamma_2),Y_t(\gamma_1))+g(t,X_t(\gamma_2)))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t}>f(t,X_t^\alpha,Y_t^{1-\alpha})+|g(t,X_t^\alpha)|\Phi^{-1}(\alpha),t\in[0,s].$$

It follows from Lemma 1 that $X_t(\gamma)>X_t^\alpha$, so $\{Y_t(\gamma_1)<Y_t^{1-\alpha}\}\cap(\Lambda_3\cap\Lambda_4)\}\subset\{X_t>X_t^\alpha\}$, and

$$\begin{aligned}\mathcal{M}\{X_s\leq X_s^\alpha\}&=1-\mathcal{M}\{X_s>X_s^\alpha\}\\&\leq 1-\mathcal{M}\{\Lambda_3\cap\Lambda_4\}=\alpha.\end{aligned} \tag{10}$$

By Eqs. (15) and (16), we have

$$\mathcal{M}\{X_s\leq X_s^\alpha\}=\alpha,$$

which means that X_s has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha)=X_s^\alpha,\quad 0<\alpha<1.$$

The theorem is verified.

Example 1. The uncertain differential equation

$$\mathrm{d}X_t=a_t\mathrm{d}t+b\mathrm{d}C_t,\quad X_0=0 \tag{11}$$

with

$$\mathrm{d}a_t=g\mathrm{d}t+h\mathrm{d}C_{1t},\quad X_0=0 \tag{12}$$

The α-path of X_t is

$$X_t^{-1}(\alpha) = a_t^\alpha \mathrm{d}t + |b|\Phi^{-1}(\alpha)t$$

where

$$a_t^\alpha = gt + |h|\Phi^{-1}(\alpha).$$

Theorem 4. *Assume that $f(t,x,y)$ is a continuous function monotone decreasing with respect to y and $g(t,x)$ is a continuous function monotone. Then the α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t, Y_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is X_t^α subject to

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha, Y_t^{1-\alpha})\mathrm{d}t + |g(t, X_t^\alpha)|\mathrm{d}C_t.$$

Then

$$\mathcal{M}\{X_t \le X_t^\alpha,\ \forall t > 0\} = \alpha, \tag{13}$$

$$\mathcal{M}\{X_t > X_t^\alpha,\ \forall t > 0\} = 1 - \alpha. \tag{14}$$

Proof: Write

$$A = \left\{t \in [0,s] | g\left(t, X_t^\alpha, Z_t^\alpha\right) \ge 0\right\},$$

$$B = \left\{t \in [0,s] | g\left(t, X_t^\alpha, Z_t^\alpha\right) < 0\right\}.$$

Then it is obvious that $A \cup B = [0,s]$. Write

$$\Lambda_1 = \left\{\gamma_2 \left| \frac{\mathrm{d}C_t(\gamma_2)}{\mathrm{d}t} \le \Phi^{-1}(\alpha) \text{ for } t \in (0,u] \right.\right\},$$

$$\Lambda_2 = \left\{\gamma_2 \left| \frac{\mathrm{d}C_t(\gamma_2)}{\mathrm{d}t} \ge \Phi^{-1}(1-\alpha) \text{ for } t \in (0,u] \right.\right\}$$

where Φ^{-1} is the inverse uncertainty distribution of $\mathcal{N}(0,1)$. Since C_t is an independent increment process, we get $\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} = \alpha$. For any $\gamma_1 \in \{Y_t(\gamma_1) \ge Y_t^{1-\alpha}\}$ and $\gamma_2 \in \Lambda_1 \cap \Lambda_2$, we have

$$f(t, X_t(\gamma_2), Y_t(\gamma_1)) + g(t, X_t(\gamma_2))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \le f(t, X_t^\alpha, Y_t^{1-\alpha}) + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), t \in [0,s].$$

It follows from Lemma 1 that $X_t(\gamma) \le X_t^\alpha$, so $\{Y_t \ge Y_t^{1-\alpha}\} \cap (\Lambda_1 \cap \Lambda_2)\} \subset \{X_t \le X_t^\alpha\}$, and

$$\mathcal{M}\{X_s \le X_s^\alpha\} \ge \mathcal{M}\{\{Y_t \ge Y_t^{1-\alpha}\} \cap (\Lambda_1 \cap \Lambda_2)\} = \alpha. \tag{15}$$

Write

$$\Lambda_3 = \left\{\gamma \left| \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > \Phi^{-1}(\alpha) \text{ for } t \in (0,u] \right.\right\},$$

$$\Lambda_4 = \left\{\gamma \left| \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} < \Phi^{-1}(1-\alpha) \text{ for } t \in (0,u] \right.\right\}.$$

Then we get $\mathcal{M}\{\Lambda_3 \cap \Lambda_4\} = 1 - \alpha$. For any $\gamma_1 \in \{Y_t(\gamma_1) > Y_t^\alpha\}$ and $\gamma_2 \in \Lambda_1 \cap \Lambda_2$, we have

$$f(t, X_t(\gamma_2), Y_t(\gamma_1)) + g(t, X_t(\gamma_2))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > f(t, X_t^\alpha, Y_t^\alpha) + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), t \in [0,s].$$

It follows from Lemma 1 that $X_t(\gamma) > X_t^\alpha$, so $\{Y_t(\gamma_1) > Y_t\} \cap (\Lambda_3 \cap \Lambda_4)\} \subset \{X_t > X_t^\alpha\}$, and

$$\begin{aligned} \mathcal{M}\{X_s \le X_s^\alpha\} &= 1 - \mathcal{M}\{X_s > X_s^\alpha\} \\ &\le 1 - \mathcal{M}\{\Lambda_3 \cap \Lambda_4\} = \alpha. \end{aligned} \tag{16}$$

By Eqs. (15) and (16), we have

$$\mathcal{M}\{X_s \le X_s^\alpha\} = \alpha,$$

which means that X_s has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = X_s^\alpha, \quad 0 < \alpha < 1.$$

The theorem is verified.

Example 2. The uncertain differential equation

$$\mathrm{d}X_t = -a_t \mathrm{d}t + b\mathrm{d}C_t, \quad X_0 = 0 \tag{17}$$

with

$$\mathrm{d}a_t = -g\mathrm{d}t + h\mathrm{d}C_{1t}, \quad X_0 = 0. \tag{18}$$

The α-path of X_t is

$$X_t^{-1}(\alpha) = -a_t^{1-\alpha}\mathrm{d}t + |b|\Phi^{-1}(\alpha)t$$

where

$$a_t^\alpha = gt + |h|\Phi^{-1}(\alpha).$$

4 Numerical Methods

Based on the previous theorem, a 99-method for solving a nested uncertain differential equation is designed as below.

Step 0: Fix a time s and set $\alpha = 0$.

Step 1: Set $\alpha \leftarrow \alpha + 0.01$.

Step 2: Solve the corresponding ordinary differential equations

$$\mathrm{d}Y_t^\alpha = f_1(t, Y_t^\alpha)\mathrm{d}t + |g_1(t, Y_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

and

$$\mathrm{d}X_t^\alpha = f(t, X_t, Y_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t,$$

respectively. Then we obtain Y_s^α and X_s^α. It is suggested to employ a numerical method to solve the equation when an analytic solution is unavailable.

Step 3: Repeat Step 1 and Step 2 for 99 times.

Step 4: The solution X_s has a 99-table,

0.01	0.02	...	0.99
$X_s^{0.01}$	$X_s^{0.02}$	...	$X_s^{0.99}$

This table gives an approximate uncertainty distribution of X_s, i.e., for any $\alpha = i/100, i = 1, 2, \cdots, 99$, we can find X_s^α from the table such that $\mathcal{M}\{X_s \leq X_s^\alpha\} = \alpha$. If $\alpha \neq i/100, i = 1, 2, \cdots, 99$, then it is suggested to employ a numerical interpolation method to get an approximate X_s^α. The 99-method can be extended to 999-method if a more precise uncertainty distribution is needed.

5 Multi-Factor Term Structure Model

The drift of term structure often displays a hump in capital market. However, the single-factor term structure models are not rich enough to fit the drift of the term structures. An uncertain reversion level grantees a better fitting for the hump in the drift structure.

The short rate is assumed following an uncertain differential equation

$$\mathrm{d}r_t = k(a + u_t - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}C_{1t} \tag{19}$$

where the parameter u_t is add as a Liu process of the drift specification. The uncertain process u_t is assumed to follow the uncertain differential equation

$$\mathrm{d}u_t = -bu_t\mathrm{d}t + \sigma_2 \mathrm{d}C_{2t}, \ u_0 = 0. \tag{20}$$

The uncertain process u_t reverts to 0 at rate b. The two canonical Liu processes C_{1t} and C_{2t} are independent. It follows from Theorem 3 that the α-path of the short rate r_t is

$$\mathrm{d}r_t^\alpha = k(a + u_t^\alpha - r_t^\alpha)\mathrm{d}t + \sigma_1 \Phi^{-1}(\alpha)\mathrm{d}t$$

where u_t^α follows

$$\mathrm{d}u_t^\alpha = -bu_t^\alpha\mathrm{d}t + \sigma_2 \Phi^{-1}(\alpha)\mathrm{d}t, \ u_0 = 0.$$

We consider the zero-coupon bond when the short rate follows the proposed nested uncertain differential equation. Denote the zero-coupon bond price as $P(T)$ with maturity date T. Following the local equilibrium hypothesis, we have

$$P(T) = E\left[\exp\left(-\int_0^T r_t \mathrm{d}t\right)\right].$$

As we have already got the α-path for the short rate r_t, then

$$\begin{aligned} P(T) &= E\left[\exp\left(-\int_0^T r_t \mathrm{d}t\right)\right] \\ &= \int_0^1 \exp\left(-\int_0^T r_t^{-1}(\alpha)\mathrm{d}t\right)\mathrm{d}\alpha \end{aligned}$$

Example 3. *Suppose that the short interest rate* r_t *follows*

$$\mathrm{d}r_t = k(a + u_t - r_t)\mathrm{d}t + \sigma_1 \mathrm{d}C_{1t},\ \ r_0 = 0.06$$

where u_t *follows*

$$\mathrm{d}u_t = -bu_t\mathrm{d}t + \sigma_2 \mathrm{d}C_{2t},\ \ u_0 = 0.02$$

Let $a = 0.05$*,* $k = 0.4$*,* $b = 0.01$*,* $\sigma_1 = \sigma_2 = 0.0025$*, then the zero-coupon bond price is* 0.9478*.*

6 Conclusions

The nested uncertain differential equations as a special type of high dimension uncertain differential equations were proposed in this paper. The concept of α-path was derived for nested uncertain differential equations. Furthermore, a numerical method for solving a nested uncertain differential equation was designed. The proposed nested uncertain differential equations were applied to build an uncertain multi-factor term structure model.

Acknowledgments

This work was supported by National Natural Science Foundation of China Grant No.61304182 and Collaborative Innovation Center for China Economy.

References

[1] X. Chen and B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optimization and Decision Making*, Vol.9, No.1, 69-81, 2010.

[2] Chen X., Uncertain calculus with finite variation processes, *Soft Computing*, Vol.19, No.10, 2905-2912, 2015.

[3] Liu B., *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[4] Liu B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, Vol.2, No.1, 3-16, 2008.

[5] Liu B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, Vol.3, No.1, 3-10, 2009.

[6] Liu B., *Theory and Practice of Uncertain Programming*, 2nd ed., Springer-Verlag, Berlin, 2009.

[7] Liu B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.

[8] Liu Y.H. and Ha M., Expected value of function of uncertain variables, *Journal of Uncertain Systems*, Vol.4, No.3, 181-186, 2010.

[9] Liu Y.H., An analytic method for solving uncertain differential equations, *Journal of Uncertain Systems*, Vol.6, No.4, 244-249, 2012.

[10] Liu Y.H., and Chen X. and Ralescu D.A., Uncertain currency model and currency option pricing, *International Journal of Intelligent Systems*, Vol.30, No.1, 40-51, 2015.

[11] Maruyama G., Continuous Markov processes and stochastic equations, *Rendiconti del Circolo Matematico di Palermo*,Vol.4, No.1, 48-90, 1955.

[12] Milstein G.N., Approximate integration of stochastic differential equations, *Theory of Probability and its Applications*, Vol.19, No.3, 557-562 1974.

[13] Peng J. and Yao K., A New Option Pricing Model for Stocks in Uncertainty Markets, *International Journal of Operations Research*, Vol.8, No.2, 18-26, 2011

[14] Rumelin W., Numerical treatment of stochastic differential equations, *SIAM Journal on Numerical Analysis*, Vol.19, No.3, 604-613, 1982.

[15] Yao K., Uncertain calculus with renewal process, *Fuzzy Optimaztion and Decision Making*, Vol.11, No.3, 285-297, 2012.

[16] Yao K. and Chen X., A Numerical method for solving uncertain differential equations, *Journal of Intelligent & Fuzzy Systems*, Vol.25, No.3, 825-832, 2013.

[17] Yao K., Gao J. and Gao Y., Some stability theorems of uncertain differential equation, *Fuzzy Optimization and Decision Making*, Vol.12, No.1, 3-13, 2013.

[18] Yao K., Multi-dimensional uncertain calculus with Liu process, *Journal of Uncertain Systems*, Vol.8, No.4, 244-254, 2014.

[19] Zhu Y., Uncertain optimal control with application to a portfolio selection model, *Cybernetics and Systems*, Vol.41, No.7, 535-547, 2010.