Uncertain Models on Railway Transportation Planning Problem

Yuan Gao, Lixing Yang, Shukai Li

State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University
Beijing 100044, China

Abstract

This paper investigates the frequency service network design problem in a railway freight transportation system, in which the fixed charge and transportation costs are both nondeterministic. In order to deal with nondeterministic system, uncertain variables are introduced. Here we propose two uncertain programming models, namely, budget-constrained model and possibility-constrained model, to design the freight transportation system. It is proved that the possibility-constrained model can be transformed to an equivalent deterministic transportation model using inverse uncertainty distribution. Based on this equivalence relation, the possibility-constrained optimal transportation plan can be obtained and then the solution of the budget-constrained model can be approximated. Finally, the idea of uncertain models is illustrated by a numerical experiment.

Keywords: Railway freight transportation; Service network design; Uncertain programming; Mixed integer linear programming

1 Introduction

In modern society, freight transportation is a vital component of the economy. A well-performed freight transportation system can ensure a low-cost and fast flow of products, which improves the economy efficiency. Due to the safety and capacity, the railway freight transportation always occupies an important place in the entire transportation system, especially for the long-distance or large-scale freight transportation. Thus, the research on how to make the railway freight transportation more efficient has lasted for decades ([1, 2, 17]).

Essentially, the railway freight transportation problem belongs to service network design problems, which aim to ensure an optimal designation and utilization of resources on networks. Crainic et al.[4] pointed out that service strategies can be divided into three planning levels, namely, strategic (long term), tactical (medium term) and operational (short term) planning level. Moreover, Crainic [5] mathematically summarized service network design problems into two categories, that is, frequency and dynamic service network design problems. In the frequency service network design problem, service frequencies are explicit integer decision variables or derived outputs in the model. Typically, frequency service network design models address issues in the strategic/tactical planning level. Dynamic service network design models, however, are closer to the operational planning level, where a time dimension is introduced into the formulation and a space-time network is used to represent the operations over time periods.

The railway freight transportation problem investigated in this paper, according to the categories mentioned above, is a special case of frequency service network design problems. In details, it focuses on a railway freight transportation problem with multiple product and multiple origins/destinations, and aims to find an optimal plan such that the total cost will be minimum. In past decades, most research

*Corresponding Author: gaoyuan@bjtu.edu.cn
paid attention to algorithms of transportation models, which can be roughly divided into two classes: the mixed-integer programming methods and heuristics methods. Theoretically, the transportation models can be formulated in the form of mixed-integer programming, and hence may be solved by the methodologies of mixed-integer programming, such as Lagrangian relaxation, branch-and-bound, cutting plane. For more details, interested readers can refer to Crainic et al.[3], Keaton[19, 20], Newton et al.[27]. The heuristics methods, rapidly developed in the past 20 years, provide a good alternative for problems with large size. In most situations, the transportation problems are so large that it is impractical to solve them precisely, and hence heuristics methods, such as Tabu Search and Genetic Algorithm, have been widely used(Gorman[15, 16], Keaton[19], Yang et al.[34, 35]).

In the literature mentioned above, however, all the parameters in transportation system, such as fixed charge, transportation costs and rail capacity, have been regarded as predetermined values. In the process of planning, however, the decision makers have to take into account all of the possible events during the planning horizon, for instance, the fluctuation of transportation costs, which is usually caused by the change of economic environment. Generally speaking, the fluctuation of cost cannot be accurately obtained, hence the decision makers have to estimate it mainly via experience. In other words, the fixed charge and transportation costs are both imprecise empirical data. In order to cope with these empirical data, the railway freight transportation models should be properly modified. In 2011, Yang et al.[34, 35] employed fuzzy variable to describe the empirical data in railway freight transportation models, in which the transportation costs and fixed charge were assumed fuzzy. In their paper, the properties of fuzzy models were analyzed and then heuristic algorithms were designed based on genetic algorithm.

Uncertainty theory, proposed in 2007([22]) and refined in 2010([25]), provides a new non-probabilistic[6, 33, 28, 36, 37] method to cope with the empirical data in optimization models. Thanks to the operational law in uncertainty theory, uncertain variable is of efficient use in monotonic system. Till now, it has appeared in inventory problems([8, 9, 13, 29]), network optimizations([10, 11, 12, 14, 18])and optimal control problems([7, 38]). In 2012, Sheng et al. [31] introduced uncertainty theory into transportation problems. They first investigated a basic model without path choice or capacity constraints, in which the supply amount, demand amount and transportation costs were regarded as uncertain variables. Later, they developed a more complex model[32], where the uncertain fixed charge was taken into account. In this paper, we will introduce the uncertainty theory into the railway freight transportation model, in which the fixed charge and transportation costs are both regarded as uncertain variables. Firstly, the total relevant cost, due to the uncertainty in fixed charge and transportation costs, is an uncertain variable, the distribution function of which will be deduced and calculated. Secondly, under additional constraints, two uncertain railway transportation models, namely, budget-constrained model and possibility-constrained model, are proposed respectively. Based on the uncertainty theory, it can be shown that the optimal solution to possibility-constrained transportation model can be obtained via solving a relevant deterministic model, which is actually another transportation problem. The optimal solution to budget-constrained transportation model, however, is more complicated to figure out. The algorithm for solving the budget-constrained transportation model is proposed after analyzing the properties of the model. Finally, a numerical experiment will be implemented to illustrate the models and algorithms.

The remainder of this paper is organized as follows. Section 2 is problem description, where uncertainty theory and frequency service network design are briefly introduced for the completeness of this research. In section 3, uncertain models for railway freight transportation problem are proposed, and the distribution function of total relevant cost is discussed. In Section 4, the properties of uncertain models are investigated and then corresponding algorithms are proposed. In Section 5, a numerical experiment is implemented to illustrate the models and algorithms. Section 6 concludes this paper with a brief summary.
2 Problem Description

2.1 Preliminaries of Uncertainty Theory

Uncertainty theory provides an axiomatic system to cope with the imprecise information in expert data. Similar to probability theory, an event is denoted by a set with a designated value, which indicates the occurrence possibility of the event. Different from that in probability theory, the occurrence possibility comes from the judgement of experts. In this section, we introduce some foundational concepts and properties of uncertainty theory and uncertain programming, which will be used throughout this paper.

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is designated a number $M(\Lambda)$. In order to ensure the set function $M(\Lambda)$ has certain mathematical properties, the following four axioms were presented([22, 23]):

**Axiom 1.** (Normality Axiom) $M(\Gamma) = 1$ for the universal set $\Gamma$.

**Axiom 2.** (Duality Axiom) $M(\Lambda_1) + M(\Lambda_2) = 1$ for any event $\Lambda_1$.

**Axiom 3.** (Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).$$

**Axiom 4.** (Product Axiom) Let $(\Gamma_i, \mathcal{L}_i, M_i)$ be uncertainty spaces for $i = 1, 2, \cdots$. Then the product uncertain measure $M$ is an uncertain measure satisfying

$$M\left(\prod_{i=1}^{\infty} \Lambda_i\right) = \prod_{i=1}^{\infty} M_i(\Lambda_i),$$

where $\Lambda_i$ are arbitrarily chosen events from $\mathcal{L}_i$ for $i = 1, 2, \cdots$, respectively.

The set function $M(\Lambda)$ is called an uncertain measure if it satisfies the first three axioms. Uncertain measure $M(\Lambda)$ is just the occurrence possibility of event $\Lambda$, which is from experts’ evaluation. Based on the axioms, the concept of uncertain variable was proposed.

**Definition 1.** (Liu [22]) An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Similar to the random variable, the distribution of an uncertain variable $\xi$ is defined by $\Phi(x) = M(\xi \leq x)$ for any real number $x$. For example, the zigzag uncertain variable $\xi \sim Z(a, b, c)$ has an uncertainty distribution $\Phi(x)$, which is shown in Figure 1.

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ (x-a)/2(b-a), & \text{if } a \leq x \leq b \\ (x+c-2b)/2(c-b), & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c. \end{cases}$$

In practices, uncertain variables are usually express linguistically, like “about 15”, “from 30 to 40, more likely 36”. Such uncertain variables satisfy some important properties both in theory and practice, which are called regular.
From the inverse distribution, we can easily obtained

\[ \Phi \]

Proof:

Let

\[ f \]

Example 1:

Theorem 1.

erational low for regular variables, which makes uncertainty theory efficient in monotonic system.

We usually assume that all uncertain variables in practical applications are regular. Otherwise, a small perturbation can be imposed to obtain a regular one. Next, we introduce an operational law for regular uncertain variables.

A real function \( f(x_1, x_2, \ldots, x_n) \) is said to be strictly increasing if \( f \) satisfies the following conditions:

1. \( f(x_1, x_2, \ldots, x_n) \geq f(y_1, y_2, \ldots, y_n) \) when \( x_i \geq y_i \) for \( i = 1, 2, \ldots, n \);
2. \( f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n) \) when \( x_i > y_i \) for \( i = 1, 2, \ldots, n \).

Given a strictly increasing function, Gao([11]) and Liu([25]) independently proved the following operational law for regular variables, which makes uncertainty theory efficient in monotonic system.

**Theorem 1.** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively, and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous and strictly increasing function. Then the uncertain variable \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) has an inverse uncertainty distribution

\[ \Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)). \]

**Example 1:** If \( h > 0, k > 0 \), we have

\[ h\mathcal{Z}(a_1, b_1, c_1) + k\mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(ha_1 + ka_2, hb_1 + kb_2, hc_1 + kc_2). \]

**Proof:** Let \( \xi_1 = \mathcal{Z}(a_1, b_1, c_1) \), \( \xi_2 = \mathcal{Z}(a_2, b_2, c_2) \) and \( \eta = h\xi_1 + k\xi_2 \). According to Theorem 1, the inverse distribution of \( \eta \) is

\[ \Psi^{-1}(\alpha) = h\Phi_1^{-1}(\alpha) + k\Phi_2^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(ha_1 + ka_2) + 2\alpha(hb_1 + kb_2), & \text{if } 0 < \alpha < 0.5 \\ (2 - 2\alpha)(hb_1 + kb_2) + (2\alpha - 1)(hc_1 + kc_2), & \text{if } 0.5 \leq \alpha < 1. \end{cases} \]

From the inverse distribution, we can easily obtained \( \eta \sim \mathcal{Z}(ha_1 + ka_2, hb_1 + kb_2, hc_1 + kc_2) \). □

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**Figure 1:** Zigzag Uncertainty Distribution

**Definition 2.** ([Liu [25]]) An uncertain variable \( \xi \) with distribution \( \Phi \) is said to be regular, if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \).

According to Definition 2, zigzag uncertain variable \( \mathcal{Z}(a, b, c) \) is a regular variable, whose inverse distribution function is \( \Phi^{-1}(\alpha) \), i.e.,

\[ \Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } 0 < \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } 0.5 \leq \alpha < 1. \end{cases} \]

Obviously, if \( \xi \) is regular, the distribution function \( \Phi(x) \) is continuous and strictly increasing at each point \( x \) with \( 0 < \Phi(x) < 1 \). Meanwhile, \( \Phi^{-1}(\alpha) \) is continuous and strictly increasing for \( \alpha \in (0, 1) \). We usually assume that all uncertain variables in practical applications are regular. Otherwise, a small perturbation can be imposed to obtain a regular one. Next, we introduce an operational law for regular uncertain variables.
2.2 Frequency Service Network Design Problem

As mentioned before, the railway freight transportation problem is a special case of frequency service network design problems. In the railway network, the stations or depots are represented by nodes, and railway links are represented by edges in the network. Thus, the railway network can be denoted by $G = (V, A)$, where $V$ is the stations/nodes set and $A$ is the links/edges set.

When the flow of products moves on railway networks, the transportation costs will arise along with fixed charges. The transportation cost, naturally, is a function related to the product type, the product amount and the characteristic of links. Usually, the cost of per unit of flow of a product on a link is predetermined, and the transportation cost of the product flow on the link is simply assumed linear function of the product amount. Besides, the fixed charges of links should be taken into account. The fixed charge of a link indicates that as soon as the link is chosen to use, no matter how many products will move along the link, a fixed cost will arise. Generally, different links have different fixed charge. Furthermore, the capacity of each link during the planning period is limited, which is mainly presented in two aspects. First, during one period, the frequency of using each link is limited. Second, on each link, the capacity of trains is limited. In this paper, we assume that on the same link, the capacity of trains is the same.

To construct the mathematical models for railway freight transportation problem, in the following sections, we need to determine the parameters, decision variables, objective function and constraints, respectively.

2.2.1 Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$V$</td>
<td>the nodes (stations) set;</td>
</tr>
<tr>
<td>$A$</td>
<td>the edges (links) set;</td>
</tr>
<tr>
<td>$i$</td>
<td>the node in the rail network, $i = 1, 2, \cdots, n \in V$;</td>
</tr>
<tr>
<td>$A_{ij}$</td>
<td>the link from node $i$ to node $j$, $(i, j) \in A$;</td>
</tr>
<tr>
<td>$K$</td>
<td>the set of products to be transported, $K = {1, 2, \cdots, k}$;</td>
</tr>
<tr>
<td>$h_{ij}$</td>
<td>the fixed charge of opening link $(i, j)$;</td>
</tr>
<tr>
<td>$w_{ij}^k$</td>
<td>the transportation cost per unit of flow of product $k$ on link $(i, j)$;</td>
</tr>
<tr>
<td>$G = (V, A, h, w)$</td>
<td>the railway network, where $h = (h_{ij})$ and $w = (w_{ij}^k)$;</td>
</tr>
<tr>
<td>$d_{ij}^k$</td>
<td>the demand of product $k$ at node $i$;</td>
</tr>
<tr>
<td>$C_{ij}$</td>
<td>the capacity of each train on link $(i, j)$;</td>
</tr>
<tr>
<td>$Z^+$</td>
<td>the set of non-negative integers;</td>
</tr>
<tr>
<td>$Z_{ij}$</td>
<td>the maximum frequency of opening link $(i, j)$, $Z_{ij} \in Z^+$;</td>
</tr>
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2.2.2 Decision variables and objective function

The transportation planning problem contains two types of decision variables:

- $(x_{ij}^k)$: amount of flow that is required for the product $k$ through the link $(i, j)$
- $(y_{ij})$: frequency that the opening link $(i, j)$ is used

For the sake of simplicity, we use $x$ and $y$ to denote the decision vectors consisting of $x_{ij}^k$ and $y_{ij}$, respectively. In this paper, we call the decision vector $(x, y)$ a transportation plan. Given a transportation plan $(x, y)$, it is easy to verify that the total relevant cost of products flow can be formulated as

$$ F(x, y) = \sum_{(i, j) \in A} h_{ij}y_{ij} + \sum_{k \in K} \sum_{(i, j) \in A} w_{ij}^kx_{ij}^k. \quad (1) $$

The first part of $F(x, y)$ is the total fixed charge of opening links, and the second part is the total transportation cost.
2.2.3 Constraints

The solution of railway freight transportation problem must satisfy the following constraints

\[ \sum_{j \in V} x^k_{ij} - \sum_{j \in V} x^k_{ji} = d^k_i, \quad i \in V, k \in K, \]  
(2)

\[ \sum_{k \in K} x^k_{ij} \leq y_{ij}C_{ij}, \quad (i, j) \in A, \]  
(3)

\[ 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, \quad (i, j) \in A. \]  
(4)

\[ x^k_{ij} \geq 0, \quad k \in K, (i, j) \in A. \]  
(5)

Equation (2) is the flow conservation condition. Specifically, the first part \( \sum_{j \in V} x^k_{ij} \) is the outflow of product \( k \) in node \( i \), and the second part \( \sum_{j \in V} x^k_{ji} \) indicates the inflow in node \( i \). If \( d^k_i > 0 \), node \( i \) is called supply node of product \( k \); if \( d^k_i < 0 \), node \( i \) is called demand node of product \( k \); if \( d^k_i = 0 \), node \( i \) is called transshipment node of product \( k \).

Constraints (3) express the capacity constraint of each link during one planning period. Constraints (4) enforce the limit on the number of transportation for the link \((i, j)\). If \( y_{ij} = 0 \), link \((i, j)\) will not be opened in the freight transportation. Note that there is a fixed charge at every opening of link \((i, j)\), the optimal transportation plan should try to lower the opening frequency.

The complete mathematical formulation of the railway freight transportation problem is written as follows

\[
\begin{align*}
\min_{x,y} & \quad F(x,y) \\
\text{s.t.} & \quad \sum_{j \in V} x^k_{ij} - \sum_{j \in V} x^k_{ji} = d^k_i, \quad i \in V, k \in K, \\
& \quad \sum_{k \in K} x^k_{ij} \leq y_{ij}C_{ij}, \quad (i, j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, \quad (i, j) \in A, \\
& \quad x^k_{ij} \geq 0, \quad k \in K, (i, j) \in A.
\end{align*}
\]  
(6)

Obviously, model (6) belongs to the mix integral linear programming (MILP), since it is assumed in this paper that the objective function and constraints are linear. The transportation plan \((x, y)\) satisfies the constraints of model (6) is called a feasible plan. Denote \( P \) as the set of feasible transportation plan of model (6). Thus, the minimum total cost on railway network \( G = (V, A, h, w) \) can be formulated as

\[ F_G = F(x^*, y^*) = \min_{(x,y) \in P} F(x, y), \]  
(7)

where feasible plan \((x^*, y^*)\) is called the optimal plan.

According to the formulation of \( F(x,y) \), it is easy to see that transportation problem belongs to the monotonic system, that is, \( F(x,y) \) is strictly increase with respect to fixed charge \( h = (h_{ij}) \) and transportation cost \( w = (w^k_{ij}), k \in K, (i, j) \in A. \) Thus, the minimum total cost \( F_G \) is also strictly increasing with respect to \( (h, w) \). More precisely, if some values of \( h_{ij} \) or \( w^k_{ij} \) increase, the minimum total cost \( F_G \) may not increase; if all the values of of \( h_{ij} \) and \( w^k_{ij} \) increase, however, the minimum total cost \( F_G \) must increase.
3 Uncertain Railway Freight Transportation Models

3.1 Uncertainty in Railway Freight Transportation

In practice, however, some parameters in model (6) can not be predetermined when planning. Generally, during the plan horizon, the fixed charge and transportation costs will fluctuate due to the change of economic environment, such as price of energy and human cost. Hence, indeterminacy should be taken into account by the decision maker.

From the point of data processing, of course, the fixed charge and transportation costs should be described by random variables, under the ideal condition that there is plenty of historical data and the probability distribution function can be obtained by statistical methods. Unfortunately, however, in many situations, there is no historical data or the economic environment has experienced such great changes that the historical data is invalid. As a result, the fluctuation of cost can only be obtained via the estimation from the decision maker, which is an imprecise empirical data.

In order to cope with the empirical data, this paper introduces uncertainty theory into the railway freight transportation model. In this paper, the fixed charge and transportation costs on link \((i, j) \in A\) of product \(k\) are considered as uncertain variables, which are denoted by \(f_{hi}^{ij}\) and \(f_{wi}^{kij}\), respectively. The distribution of \(f_{hi}^{ij}\) is denoted by \(\Psi_{ij}\), and the distribution of \(f_{wi}^{kij}\) is denoted by \(\Phi_{kij}\).

We use \(G = (\mathcal{V}, A; e_h, e_w)\) to denote the uncertain railway network with uncertain cost coefficients. Thus, given a transportation plan \((x, y)\), the total relevant cost is modified as

\[
\tilde{F}(x, y) = \sum_{(i, j) \in A} h_{ij}y_{ij} + \sum_{k \in K} \sum_{(i, j) \in A} w_{ij}^{k}x_{ij}^k.
\]

(8)

Obviously, the total cost \(\tilde{F}(x, y)\) obtained in formula (8) is an uncertain variable. The uncertainty distribution of \(\tilde{F}(x, y)\) is denoted by \(\Gamma_{xy}\), i.e.,

\[
\Gamma_{xy}(l) = M\{\tilde{F}(x, y) \leq l\}.
\]

Since both \(w_{ij}^{k}\) and \(h_{ij}\) have regular uncertainty distribution, according to Theorem 1, \(\Gamma_{xy}(l)\) is also regular. More precisely, for any \(\alpha \in (0, 1)\),

\[
\Gamma_{xy}^{-1}(\alpha) = \sum_{(i, j) \in A} \Psi_{ij}^{-1}(\alpha)y_{ij} + \sum_{k \in K} \sum_{(i, j) \in A} \Phi_{kij}^{-1}(\alpha)x_{ij}^k.
\]

Then, we can obtain the distribution function \(\Gamma_{xy}(l)\) via its inverse function \(\Gamma_{xy}^{-1}(\alpha)\).

It is easy to see that in uncertain environment the total cost,

\[
\bar{F}_G = \min_{(x, y) \in P} \tilde{F}(x, y),
\]

(9)

is also an uncertain variable with uncertainty distribution \(\Gamma_G\), i.e.,

\[
\Gamma_G(l) = M\{\bar{F}_G \leq l\}.
\]

Similarly, \(\Gamma_G(l)\) is also a regular distribution.

For any feasible plan \((x, y)\), according to formula (9), it is easy to see

\[
\bar{F}_G \leq \tilde{F}(x, y),
\]

thus,

\[
M\{\bar{F}_G \leq l\} \geq M\{\tilde{F}(x, y) \leq l\},
\]

7
obtain the value of $\Gamma^{-1}_G\alpha$.

Since $\Gamma^{-1}_G\alpha$ and $\Gamma_{xy}(l)$ are both regular, for any $\alpha \in (0, 1)$, we have the following inequality

$$\Gamma^{-1}_G\alpha \leq \Gamma^{-1}_{xy}(l).$$

(10)

### 3.2 Distribution Function of $F_G$

Although inequality (10) gives the upper bound of $\Gamma^{-1}_G\alpha$, we want to go further, more precisely, to obtain the value of $\Gamma^{-1}_G\alpha$.

As mentioned before, in deterministic railway network $G = (V, A, h, w)$, optimal total cost $F_G$ is strictly increasing with respect to $(h, w)$. For the sake of convenience, $F_G$ can be formulated by

$$F_G = f(h, w),$$

which indicates that once $(h, w)$ is determined, the optimal total cost $f(h, w)$ can then be obtained by model (6), i.e.,

$$f(h, w) = F(x^*, y^*),$$

where $(x^*, y^*)$ is the optimal solution of model (6).

According to Theorem 1, for any $\alpha \in (0, 1)$, we can obtain $\Gamma^{-1}_G\alpha$ by

$$\Gamma^{-1}_G\alpha = f(\Psi^{-1}(\alpha), \Phi^{-1}(\alpha)),$$

where $\Psi^{-1}(\alpha) = (\Psi^{-1}(\alpha)_{ij})$ and $\Phi^{-1}(\alpha) = (\Phi^{-1}(\alpha)_{kij})$. The value of $f(\Psi^{-1}(\alpha), \Phi^{-1}(\alpha))$ is just the optimal objective of following model

$$\min_{x, y} f(\Psi^{-1}(\alpha), \Phi^{-1}(\alpha)) = \sum_{(i, j) \in A} \Psi^{-1}(\alpha)_{ij}y_{ij} + \sum_{k \in K} \sum_{(i, j) \in A} \Phi^{-1}(\alpha)_{kij}x_{ij}^k$$

s.t. \quad \begin{align*}
\sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k &= d_i^k, i \in V, k \in K, \\
\sum_{k \in K} x_{ij}^k &\leq y_{ij}C_{ij}, (i, j) \in A, \\
0 &\leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i, j) \in A, \\
x_{ij}^k &\geq 0, k \in K, (i, j) \in A.
\end{align*}

(11)

Simply, to obtain the value of $\Gamma^{-1}_G\alpha$, we should first calculate the values of $\Psi^{-1}(\alpha)_{ij}$ and $\Phi^{-1}(\alpha)_{kij}$, $(i, j) \in A, k \in K$. Then, solve model (11), whose optimal objective is just the value of $\Gamma^{-1}_G\alpha$.

Usually, to obtain the distribution function $\Gamma_G(l)$, we may use the 99-method.

Algorithm 1. (99-method)

Step 1: Set $\alpha = 0.01$.

Step 2: Calculate the values of $\Psi^{-1}(\alpha)_{ij}$ and $\Phi^{-1}(\alpha)_{kij}$.

Step 3: Solve model (11), whose optimal objective is denoted by $\Gamma^{-1}_G\alpha$. Save $(\Gamma^{-1}_G\alpha, \alpha)$.

Step 4: If $\alpha = 0.99$, stop; otherwise, set $\alpha = \alpha + 0.01$ and go to Step 2.

Via the 99-method, we obtain 99 points $(\Gamma^{-1}_G\alpha, \alpha)$, which satisfies

$$\Gamma_G(\Gamma^{-1}_G\alpha) = \alpha.$$

Then, the distribution function $\Gamma_G(l)$ can be approximated by interpolation.
In deterministic railway network \( G = (V,A,h,w) \), \( F_G \) will be the total cost if the optimal transportation plan \((x,y)\) is implemented. In uncertain railway network \( \tilde{G} = (V,A,h,\tilde{w}) \), however, \( \tilde{F}_G \) loses this meaning. Later, we will illustrate that the significance of both \( \tilde{F}_G \) and \( \Gamma \) is only in theory but not in application, since the transportation model (6) will be modified.

3.3 Budge-constrained Optimal Plan and Possibility-constrained Optimal Plan

In order to modify the original model (6), some additional criteria should be considered. In practice, a budget level is usually given in advance. Naturally, the decision makers try to control the total cost lower than the budget level with the greatest possibility. With that in mind, the concept of budget-constrained optimal transportation plan is proposed in the following definition.

Definition 3. Given budget level \( L_0 \), the feasible solution \((x^*,y^*)\) is called the budget-constrained optimal transportation plan if \((x^*,y^*)\) satisfies

\[
\Gamma_{x^*y^*}(L_0) = \max_{(x,y) \in P} \Gamma_{xy}(L_0),
\]

i.e.,

\[
M\left\{ \tilde{F}(x^*,y^*) \leq L_0 \right\} = \max_{(x,y) \in P} M\left\{ \tilde{F}(x,y) \leq L_0 \right\},
\]

where \( P \) is the set of feasible solution of model (6).

For any plan \((x,y)\), it corresponds to a total cost \( \tilde{F}(x,y) \), which is an uncertain variable. Given budget level \( L_0 \), the value \( \Gamma_{xy}(L_0) = M\{\tilde{F}(x,y) \leq L_0\} \) describes the possibility that the total cost \( \tilde{F}(x,y) \) is lower than budget level \( L_0 \). Simply, among all the plan \((x,y)\), the budget-constrained optimal transportation plan \((x^*,y^*)\) corresponds to an uncertain total cost \( \tilde{F}(x^*,y^*) \) which is less than budget level \( L_0 \) with the greatest possibility. Hence, the original railway freight transportation model can be modified as

\[
\begin{align*}
\max_{x,y} & \quad M\left\{ \tilde{F}(x,y) \leq L_0 \right\} \\
\text{s.t.} & \quad \sum_{i \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = d_i^k, i \in V, k \in K, \\
& \quad \sum_{k \in K} x_{ij}^k \leq y_{ij} c_{ij}, (i,j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i,j) \in A, \\
& \quad x_{ij}^k \geq 0, k \in K, (i,j) \in A.
\end{align*}
\]

(12)

It is easy to see that the constraints are the same to those of the original model (6). In other words, the budget-constrained transportation model shares the same feasible plan set \( P \) with the original transportation model (6). According to Definition 3, the solution of model (12) is the budget-constrained optimal transportation plan \((x^*,y^*)\).

Contrary to the situation of budget-constrained optimal transportation plan, sometimes the decision makers need to determine a budget level \( L_0 \), such that there exists a transportation plan \((x,y)\) satisfying \( M\{\tilde{F}(x,y) \leq L_0\} \geq \alpha \), where \( \alpha \) is a predetermined possibility level. For instance, given \( \alpha = 0.9 \), the decision makers have to determine a budget level \( L_0 \) and then choose a plan \((x,y)\) satisfying \( M\{\tilde{F}(x,y) \leq L_0\} \geq 0.9 \). That is, if the transportation firm chooses plan \((x,y)\), the total cost will be lower than \( L_0 \) with possibility at least 90%. Of course, for a given possibility level \( \alpha \), the decision maker always want to determine a low enough budget level \( L_0 \) and corresponding transportation plan \((x,y)\). Hence, the concept of possibility-constrained optimal transportation plan is proposed.
Definition 4. Given possibility level $\alpha$, the feasible solution $(x^*, y^*)$ is called the possibility-constrained optimal transportation plan if for any feasible solution $(x, y)$, plan $(x^*, y^*)$ satisfies

$$L_0 = \min \{ L | \Gamma_{x^*, y^*}(L) \geq \alpha \} \leq \min \{ L | \Gamma_{xy}(L) \geq \alpha \},$$

i.e.,

$$L_0 = \min \{ L | M \{ \tilde{F}(x^*, y^*) \leq L \} \geq \alpha \} \leq \min \{ L | M \{ \tilde{F}(x, y) \leq L \} \geq \alpha \}.$$

Under the idea of Definition 4, the original railway freight transportation model can be modified as

$$\begin{aligned}
\min_{x,y} & \quad L_0 \\
\text{s.t.} & \quad \Gamma_{xy}(L_0) = M \{ \tilde{F}(x, y) \leq L_0 \} \geq \alpha, \\
& \quad \sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = d_i^k, i \in V, k \in K, \\
& \quad \sum_{k \in K} x_{ij}^k \leq y_{ij} C_{ij}, (i, j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i, j) \in A, \\
& \quad x_{ij}^k \geq 0, k \in K, (i, j) \in A.
\end{aligned} \tag{13}$$

The optimal solution of model (13) is just the possibility-constrained optimal transportation plan.

In uncertain environment, no matter the budget-constrained model or the possibility-constrained model, it is difficult to search the exact optimal solution. In most literature on uncertain programming, heuristic algorithms, such as genetic algorithm and tuba search, were adopted to find the near optimal solution of uncertain models. Although the effectiveness and efficiency of these methods is always illustrated by numerical experiments, the rigor and portability of these methods are not satisfying. Fortunately, in the framework of uncertainty theory, these two uncertain models can both be transformed to corresponding deterministic railway freight transportation problems. As a result, the well researched methods in traditional deterministic railway freight transportation problem can be employed.

4 Solution of Uncertain Models

In uncertain railway network $\tilde{G} = (V, A, \tilde{h}, \tilde{w})$, assume that the transportation cost $w_{ij}^k$ are regular uncertain variables with distribution $\Phi_{kij}$, $k \in K, (i, j) \in A$, and the fixed charge $h_{ij}$ are also regular uncertain variables with distribution $\Psi_{ij}$, $(i, j) \in A$. We find that the there exists an equivalent relationship between the optimal solution to possibility-constrained model and that to deterministic transportation model, i.e.,

**Theorem 2.** Given possibility level $\alpha$, the possibility-constrained optimal plan $(x^*, y^*)$ is just the optimal plan of deterministic railway network $G' = (V', A', h', w')$, where $V' = V, A' = A$ and $h_{ij}' = \Psi_{ij}^{-1}(\alpha)$, $w_{ij}^k = \Phi_{kij}^{-1}(\alpha)$.

**Proof:** According to Definition 4, the optimal possibility-constrained plan $(x^*, y^*)$ is the optimal
solution to the following model:

\[
\begin{aligned}
\min_{x,y} & \quad L_0 \\
\text{s.t.} & \quad \Gamma_{xy}(L_0) = M \left\{ \bar{F}(x,y) \leq L_0 \right\} \geq \alpha, \\
& \quad \sum_{j \in V} x^k_{ij} - \sum_{j \in V} x^k_{ji} = d^k_i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x^k_{ij} \leq y_{ij} C_{ij}, (i,j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i,j) \in A, \\
& \quad x^k_{ij} \geq 0, k \in K, (i,j) \in A.
\end{aligned}
\]

By comparison with model (6), the optimal solution to model (16) is just the optimal transportation plan

\[
\begin{aligned}
\sum_{(i,j) \in A} \Psi^{-1}(\alpha) y_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi^{-1}(\alpha) x^k_{ij} \leq L_0.
\end{aligned}
\]

Thus, model (13) can be equivalently transformed to the following deterministic model:

\[
\begin{aligned}
\min_{x,y} & \quad L_0 \\
\text{s.t.} & \quad \sum_{(i,j) \in A} \Psi^{-1}(\alpha) y_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi^{-1}(\alpha) x^k_{ij} \leq L_0, \\
& \quad \sum_{j \in V} x^k_{ij} - \sum_{j \in V} x^k_{ji} = d^k_i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x^k_{ij} \leq y_{ij} C_{ij}, (i,j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i,j) \in A, \\
& \quad x^k_{ij} \geq 0, k \in K, (i,j) \in A.
\end{aligned}
\]

It is easy to see that model (15) is equivalent to the following model

\[
\begin{aligned}
\min_{x,y} & \quad \sum_{(i,j) \in A} \Psi^{-1}(\alpha) y_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi^{-1}(\alpha) x^k_{ij} \\
\text{s.t.} & \quad \sum_{j \in V} x^k_{ij} - \sum_{j \in V} x^k_{ji} = d^k_i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x^k_{ij} \leq y_{ij} C_{ij}, (i,j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i,j) \in A, \\
& \quad x^k_{ij} \geq 0, k \in K, (i,j) \in A.
\end{aligned}
\]

By comparison with model (6), the optimal solution to model (16) is just the optimal transportation plan of the deterministic railway network \( G' = (V', A', h', w') \), where \( V' = V, A' = A, \) and \( h'_{ij} = \Psi^{-1}(\alpha), w'^k_{ij} = \Phi^{-1}(\alpha) \). The theorem is proved. \( \square \)

As Theorem 2 illustrates, to solve the possibility-constrained model (13), we first construct a deterministic railway network \( G' = (V', A', h', w') \), where \( V' = V, A' = A, \) and \( h'_{ij} = \Psi^{-1}(\alpha), w'^k_{ij} = \Phi^{-1}(\alpha) \). Then, we can adapt the well-researched methods to solve the deterministic railway transportation problem in \( G' = (V', A', h', w') \), which is essentially a MILP problem. In this paper, we use CPLEX to solve this MILP problem, implemented through the cplex interface function of Matlab TOMLAB toolbox.

For the budget-constrained model (12), we also prove that it can be transformed to an equivalent deterministic model.
Theorem 3. Given budget level \( L_0 \), the budget-constrained optimal plan \((x^*, y^*)\) is just the possibility-constrained optimal plan of uncertain railway network \( G = (V, A, \tilde{h}, \tilde{w}) \) with possibility level \( \alpha \), where \( \alpha = \Gamma^*_G(L_0) \).

Proof: According to Definition 4, the possibility-constrained optimal plan \((x^*, y^*)\) is the optimal solution to the following model:

\[
\begin{align*}
\max_{x,y} & \quad M \left\{ \tilde{F}(x, y) \leq L_0 \right\} \\
\text{s.t.} & \quad \sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = d_k^i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x_{ij}^k \leq y_{ij}C_{ij}, (i, j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i, j) \in A, \\
& \quad x_{ij}^k \geq 0, k \in K, (i, j) \in A.
\end{align*}
\]

Set \( \alpha = M\{\tilde{F}(x, y) \leq L_0\} \), then the above model is equivalent to the following model:

\[
\begin{align*}
\max_{x,y,\alpha} & \quad \alpha \\
\text{s.t.} & \quad \Gamma_{xy}(L_0) = M \left\{ \tilde{F}(x, y) \leq L_0 \right\} \geq \alpha, \\
& \quad \sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = d_k^i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x_{ij}^k \leq y_{ij}C_{ij}, (i, j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i, j) \in A, \\
& \quad x_{ij}^k \geq 0, k \in K, (i, j) \in A.
\end{align*}
\]

The solution of model (17) can be written as \((x, y, \alpha)\). Since \( \Gamma_{xy} \) is a regular uncertainty distribution, as the proof in Theorem 3, the first constraint of model (17) can be reformulated as \( \Gamma_{xy}^{-1}(\alpha) \leq L_0 \), i.e.,

\[
\sum_{(i,j) \in A} \Phi_{ij}^{-1}(\alpha)y_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi_{ij}^{-1}(\alpha)x_{ij}^k \leq L_0.
\]

Thus, model (17) can be reformulated as

\[
\begin{align*}
\max_{x,y,\alpha} & \quad \alpha \\
\text{s.t.} & \quad \sum_{(i,j) \in A} \Psi_{ij}^{-1}(\alpha)y_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi_{ij}^{-1}(\alpha)x_{ij}^k \leq L_0, \\
& \quad \sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = d_k^i, i \in V, k \in K, \\
& \quad \sum_{k \in K} x_{ij}^k \leq y_{ij}C_{ij}, (i, j) \in A, \\
& \quad 0 \leq y_{ij} \leq Z_{ij}, y_{ij} \in Z^+, (i, j) \in A, \\
& \quad x_{ij}^k \geq 0, k \in K, (i, j) \in A.
\end{align*}
\]

Set \( \tilde{\alpha} = \Gamma^*_\tilde{G}(L_0) = M\{\tilde{F}_{\tilde{G}} \leq L_0\} \). Since \( \Gamma_{\tilde{G}} \) is a regular distribution, we have \( \Gamma_{\tilde{G}}^{-1}(\tilde{\alpha}) = L_0 \). For the possibility level \( \tilde{\alpha} \), according to Theorem 2, we can obtain possibility-constrained optimal transportation plan \((\tilde{x}, \tilde{y})\) of railway network \( \tilde{G} = (V, A, \tilde{h}, \tilde{w}) \). In other words, \((\tilde{x}, \tilde{y})\) is the optimal solution of model (15).
That is, \((\bar{x}, \bar{y})\) and \(\bar{\alpha}\) satisfy the following constraints

\[
\begin{align*}
\sum_{(i,j) \in A} \Psi_{ij}^{-1}(\bar{\alpha}) \bar{y}_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi_{kij}^{-1}(\bar{\alpha}) \bar{x}_{ij}^k & \leq L_0, \\
\sum_{j \in V} \bar{x}_{ij}^k - \sum_{j \in V} \bar{x}_{ji}^k & = d_i^k, i \in V, k \in K, \\
\sum_{k \in K} \bar{x}_{ij}^k & \leq \bar{y}_{ij}^k C_{ij}, (i, j) \in A, \\
0 & \leq \bar{y}_{ij} \leq Z_{ij}, \bar{y}_{ij} \in \mathbb{Z}^+, (i, j) \in A,
\end{align*}
\]

The above formula indicates that \((\bar{x}, \bar{y}, \bar{\alpha})\) is a feasible solution to model (18), and \(\bar{\alpha}\) is the corresponding value of the objective function. We will prove that \((\bar{x}, \bar{y}, \bar{\alpha})\) is also the optimal solution of model (18).

Assume that \((\hat{x}, \hat{y}, \hat{\alpha})\) is a feasible solution of model (18), and \(\hat{\alpha}\) is the corresponding value of the objective function, which satisfies \(\hat{\alpha} > \bar{\alpha}\).

It is known that \(\Gamma_G^{-1}\) is a regular uncertainty distribution, which means that the inverse distribution \(\Gamma_G^{-1}(\alpha)\) is strictly increasing, i.e.,

\[
\Gamma_G^{-1}(\hat{\alpha}) > \Gamma_G^{-1}(\bar{\alpha}) = L_0.
\]

According to inequality (10), for transportation plan \((\hat{x}, \hat{y})\), we have

\[
\Gamma_G^{-1}(\hat{\alpha}) \geq \Gamma_G^{-1}(\bar{\alpha}) > L_0.
\]

More precisely,

\[
\sum_{(i,j) \in A} \Psi_{ij}^{-1}(\bar{\alpha}) \hat{y}_{ij} + \sum_{k \in K} \sum_{(i,j) \in A} \Phi_{kij}^{-1}(\bar{\alpha}) \hat{x}_{ij}^k = \Gamma_G^{-1}(\hat{\alpha}) \geq \Gamma_G^{-1}(\bar{\alpha}) > L_0.
\]

(19)

Obviously, inequality (19) leads to a contradiction with the constraints in model (18). That is, the assumption is wrong. Thus, \(\bar{\alpha} = \Gamma_G(L_0)\) is the maximum value of the objective function, and then the feasible solution \((\bar{x}, \bar{y})\) is the optimal solution of model (18). The theorem is proved.

In uncertain railway network \(\bar{G} = (V, A, \bar{h}, \bar{w})\), given budget level \(L_0\), to solve the budget-constrained model (12), we propose the following algorithm

**Algorithm 2.** *(Algorithm for budget-constrained optimal transportation plan)*

**Step 1:** Via the 99-method and interpolation method, obtain the distribution function \(\Gamma_{\bar{G}}(l)\).

**Step 2:** Set \(\alpha = \Gamma_{\bar{G}}(L_0)\).

**Step 3:** Search the possibility-constrained optimal transportation plan under possibility level \(\alpha\).

According to Theorem 3, in Step 3, the possibility-constrained optimal transportation plan under possibility level \(\alpha = \Gamma_{\bar{G}}(L_0)\) is just the budget-constrained optimal transportation plan we want.

## 5 Numerical Experiment

In this section, a numerical example of uncertain railway transportation problem if presented to show the efficiency of the models and algorithms. We consider an uncertain network \(\bar{G} = (V, A, \bar{h}, \bar{w})\) with 12 stations and 19 links, depicted in Figure 2.
To construct the mathematical model, the parameters are listed in Table 1. Two ODs will be considered in network $\tilde{G}$, that is, product 1 will be transported from station 2 to station 12 with the amount of 265 units, while product 2 will be transported from station 3 to station 11 with the amount of 195 units.

First, possibility-constrained model. Given $\alpha = 0.95$, we want to search the possibility-constrained optimal plan. As Theorem 2 indicates, we should first construct a deterministic network $G' = (V', A', \tilde{h}', \tilde{w}')$, that is, $V' = V, A' = A$ and $h'_{ij} = \Psi_{ij}^{-1}(0.95), w'_{kj} = \Phi_{kj}^{-1}(0.95)$. Then, the possibility-constrained optimal plan is just the optimal plan in network $G' = (V', A', \tilde{h}', \tilde{w}')$. According to model (6), we can obtain following transportation plan through the cplex interface function of Matlab TOMLAB toolbox, i.e., the total relevant cost is 234950.6, and the corresponding optimal paths of this transportation plan are shown in the following table.

Table 1: The parameters in numerical experiment

<table>
<thead>
<tr>
<th>Index</th>
<th>Link $(i, j)$</th>
<th>Capacity</th>
<th>Max Frequency</th>
<th>Fixed charge</th>
<th>Cost P1</th>
<th>Cost P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 2)</td>
<td>11</td>
<td>25</td>
<td>(25, 30, 32)</td>
<td>(20, 25, 27)</td>
<td>(50, 52, 58)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 3)</td>
<td>10</td>
<td>30</td>
<td>(20, 24, 26)</td>
<td>(28, 30, 34)</td>
<td>(52, 55, 60)</td>
</tr>
<tr>
<td>3</td>
<td>(1, 4)</td>
<td>5</td>
<td>10</td>
<td>(9, 15, 19)</td>
<td>(18, 20, 24)</td>
<td>(38, 42, 48)</td>
</tr>
<tr>
<td>4</td>
<td>(2, 4)</td>
<td>5</td>
<td>15</td>
<td>(9, 14, 16)</td>
<td>(45, 48, 56)</td>
<td>(90, 100, 115)</td>
</tr>
<tr>
<td>5</td>
<td>(2, 6)</td>
<td>4</td>
<td>40</td>
<td>(11, 15, 17)</td>
<td>(90, 100, 105)</td>
<td>(135, 150, 170)</td>
</tr>
<tr>
<td>6</td>
<td>(2, 8)</td>
<td>6</td>
<td>30</td>
<td>(13, 15, 18)</td>
<td>(90, 110, 125)</td>
<td>(200, 210, 230)</td>
</tr>
<tr>
<td>7</td>
<td>(3, 4)</td>
<td>7</td>
<td>30</td>
<td>(15, 20, 23)</td>
<td>(23, 25, 28)</td>
<td>(50, 52, 57)</td>
</tr>
<tr>
<td>8</td>
<td>(3, 5)</td>
<td>3</td>
<td>35</td>
<td>(27, 31, 33)</td>
<td>(58, 61, 69)</td>
<td>(100, 112, 120)</td>
</tr>
<tr>
<td>9</td>
<td>(3, 7)</td>
<td>6</td>
<td>20</td>
<td>(27, 30, 31)</td>
<td>(55, 60, 70)</td>
<td>(105, 110, 125)</td>
</tr>
<tr>
<td>10</td>
<td>(4, 7)</td>
<td>10</td>
<td>30</td>
<td>(25, 30, 32)</td>
<td>(69, 70, 75)</td>
<td>(145, 150, 165)</td>
</tr>
<tr>
<td>11</td>
<td>(5, 10)</td>
<td>5</td>
<td>25</td>
<td>(15, 20, 22)</td>
<td>(93, 100, 110)</td>
<td>(210, 220, 230)</td>
</tr>
<tr>
<td>12</td>
<td>(6, 9)</td>
<td>5</td>
<td>20</td>
<td>(10, 15, 17)</td>
<td>(113, 120, 130)</td>
<td>(240, 255, 275)</td>
</tr>
<tr>
<td>13</td>
<td>(7, 9)</td>
<td>6</td>
<td>35</td>
<td>(15, 18, 21)</td>
<td>(67, 70, 75)</td>
<td>(130, 135, 145)</td>
</tr>
<tr>
<td>14</td>
<td>(7, 10)</td>
<td>9</td>
<td>35</td>
<td>(20, 27, 30)</td>
<td>(66, 70, 80)</td>
<td>(150, 160, 180)</td>
</tr>
<tr>
<td>15</td>
<td>(8, 9)</td>
<td>4</td>
<td>25</td>
<td>(15, 18, 20)</td>
<td>(96, 100, 110)</td>
<td>(145, 150, 165)</td>
</tr>
<tr>
<td>16</td>
<td>(8, 11)</td>
<td>4</td>
<td>20</td>
<td>(8, 10, 12)</td>
<td>(77, 80, 90)</td>
<td>(150, 160, 175)</td>
</tr>
<tr>
<td>17</td>
<td>(9, 12)</td>
<td>10</td>
<td>15</td>
<td>(36, 41, 43)</td>
<td>(110, 120, 135)</td>
<td>(190, 200, 215)</td>
</tr>
<tr>
<td>18</td>
<td>(10, 12)</td>
<td>13</td>
<td>20</td>
<td>(40, 45, 55)</td>
<td>(115, 120, 135)</td>
<td>(185, 200, 225)</td>
</tr>
<tr>
<td>19</td>
<td>(11, 12)</td>
<td>8</td>
<td>30</td>
<td>(36, 45, 50)</td>
<td>(83, 90, 100)</td>
<td>(190, 200, 220)</td>
</tr>
</tbody>
</table>
Table 2: Paths of possibility-constrained optimal plan when $\alpha = 0.95$

<table>
<thead>
<tr>
<th>OD \ Product</th>
<th>Amount</th>
<th>Path</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>2 $\rightarrow$ 4 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>90</td>
<td>2 $\rightarrow$ 6 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>2 $\rightarrow$ 8 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
<td>2 $\rightarrow$ 8 $\rightarrow$ 9 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>3 $\rightarrow$ 1 $\rightarrow$ 2 $\rightarrow$ 8 $\rightarrow$ 11</td>
</tr>
<tr>
<td>2</td>
<td>115</td>
<td>3 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12 $\rightarrow$ 11</td>
</tr>
</tbody>
</table>

That is, if we choose the paths and amount in Table 2, the total relevant cost will be less than 234950.6 with possibility 95%. Moreover, 234950.6 is the minimum total relevant cost that can be achieved under the possibility constraint $\alpha = 0.95$. Simultaneously, the use frequency of corresponding links can be obtained, i.e.

Table 3: Frequency of links when $\alpha = 0.95$

<table>
<thead>
<tr>
<th>Link (i,j)</th>
<th>Frequency</th>
<th>Link (i,j)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>8</td>
<td>(2,4)</td>
<td>15</td>
</tr>
<tr>
<td>(2,6)</td>
<td>23</td>
<td>(2,8)</td>
<td>30</td>
</tr>
<tr>
<td>(3,1)</td>
<td>8</td>
<td>(3,7)</td>
<td>20</td>
</tr>
<tr>
<td>(4,7)</td>
<td>8</td>
<td>(6,9)</td>
<td>18</td>
</tr>
<tr>
<td>(7,10)</td>
<td>26</td>
<td>(8,9)</td>
<td>25</td>
</tr>
<tr>
<td>(8,11)</td>
<td>20</td>
<td>(9,7)</td>
<td>7</td>
</tr>
<tr>
<td>(9,12)</td>
<td>15</td>
<td>(10,12)</td>
<td>18</td>
</tr>
<tr>
<td>(12,11)</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Given $\alpha = 0.75$, by the same method, we can obtain the possibility-constrained optimal plan shown in Table 4.

Table 4: Paths of possibility-constrained optimal plan when $\alpha = 0.75$

<table>
<thead>
<tr>
<th>OD \ Product</th>
<th>Amount</th>
<th>Path</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>2 $\rightarrow$ 4 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>85</td>
<td>2 $\rightarrow$ 6 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>65</td>
<td>2 $\rightarrow$ 8 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>35</td>
<td>2 $\rightarrow$ 8 $\rightarrow$ 9 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>2 $\rightarrow$ 8 $\rightarrow$ 11 $\rightarrow$ 12</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
<td>3 $\rightarrow$ 1 $\rightarrow$ 2 $\rightarrow$ 8 $\rightarrow$ 11</td>
</tr>
<tr>
<td>2</td>
<td>120</td>
<td>3 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12 $\rightarrow$ 11</td>
</tr>
</tbody>
</table>

In this case, the total relevant cost is 225467. That is, if we choose the paths and amount in Table 4, the total relevant cost will be less than 225467 with possibility 75%. Moreover, 225467 is the minimum total relevant cost that can be achieved under the possibility constraint $\alpha = 0.75$. The use frequency of corresponding links is listed in Table 5.
Table 5: Frequency of links when $\alpha = 0.75$

<table>
<thead>
<tr>
<th>Link (i,j)</th>
<th>Frequency</th>
<th>Link (i,j)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>7</td>
<td>(2,4)</td>
<td>15</td>
</tr>
<tr>
<td>(2,6)</td>
<td>23</td>
<td>(2,8)</td>
<td>30</td>
</tr>
<tr>
<td>(3,1)</td>
<td>8</td>
<td>(3,7)</td>
<td>20</td>
</tr>
<tr>
<td>(4,7)</td>
<td>8</td>
<td>(6,9)</td>
<td>17</td>
</tr>
<tr>
<td>(7,10)</td>
<td>26</td>
<td>(8,9)</td>
<td>25</td>
</tr>
<tr>
<td>(8,11)</td>
<td>20</td>
<td>(9,7)</td>
<td>6</td>
</tr>
<tr>
<td>(9,12)</td>
<td>15</td>
<td>(10,12)</td>
<td>18</td>
</tr>
<tr>
<td>(11,12)</td>
<td>1</td>
<td>(12,11)</td>
<td>15</td>
</tr>
</tbody>
</table>

Obviously, as the possibility $\alpha$ increases, the total relevant cost will increase. It is because the greater $\alpha$ is, the more conservative the plan is, which leads to greater cost.

Second, for uncertain railway network $\tilde{G} = (V, A, \tilde{h}, \tilde{w})$, we can obtain the distribution function $\Gamma_{\tilde{G}}$ of total relevant cost $\tilde{F}_G$ by employing the Algorithm 1. See Figure 3, by the definition of Zigzag variable, $\tilde{F}_G$ is just a Zigzag variable. Here we consider the variables are zigzag uncertain variables, so $\tilde{F}_G$ is also a zigzag uncertain variable.

Third, obtaining the distribution function $\Gamma_{\tilde{G}}$, we can further solve the budget-constrained model. Given budget $L_0 = 23000$, to obtain the budget-constrained optimal transportation plan, we should first calculate $\alpha = \Gamma_{\tilde{G}}(L_0)$ according to Theorem 3. According to the result of Algorithm 1, $\Gamma_{\tilde{G}}(23000) \approx 0.85$, that is, $\alpha = 0.85$. Then the budget-constrained optimal plan we want to obtain is just the possibility-constrained optimal plan under possibility constraint $\alpha = 0.85$. Then, by Theorem 2, we obtain the transportation plan listed in Table 6 and Table 7, which are the same to Table 2 and Table 3 respectively.

Table 6: Paths of budget-constrained optimal plan when $L_0 = 23000$

<table>
<thead>
<tr>
<th>OD</th>
<th>Product</th>
<th>Amount</th>
<th>Path</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>2</td>
<td>4 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>90</td>
<td>2</td>
<td>6 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>2</td>
<td>8 $\rightarrow$ 9 $\rightarrow$ 12</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
<td>2</td>
<td>8 $\rightarrow$ 9 $\rightarrow$ 7 $\rightarrow$ 10 $\rightarrow$ 12</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>3</td>
<td>1 $\rightarrow$ 2 $\rightarrow$ 8 $\rightarrow$ 11</td>
</tr>
<tr>
<td>2</td>
<td>115</td>
<td>3</td>
<td>7 $\rightarrow$ 10 $\rightarrow$ 12 $\rightarrow$ 11</td>
</tr>
</tbody>
</table>
If the transportation plan in Table 6 is chosen, the total relevant cost will be less than budget constraint $L_0 = 23000$ with the greatest possibility, i.e., 85%.

6 Conclusion

In practice, due to the fluctuation of cost, the deterministic railway freight transportation models can no longer efficiently work. Usually, the fluctuation of cost can not be accurately obtained, but be estimated via experience, that is, the fixed charge and transportation costs are both imprecise empirical data. In order to deal with these empirical data in railway networks, we introduced uncertain variables to describe transportation costs and fixed charge in railway freight transportation model.

Under the framework of uncertainty theory, the distribution function of the total relevant cost of the railway transportation is investigated, and an algorithm is proposed to calculate the distribution function. Then, under specific constraints, budget-constrained model and possibility-constrained model are proposed respectively. Based on the uncertainty theory, it proves that the optimal solution to possibility-constrained transportation model is equivalent to a deterministic model, which is actually another transportation problem. The algorithm for solving the budget-constrained transportation model is also proposed. Finally, a numerical experiment is implemented to illustrate the models and algorithms.

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