Law of large numbers for uncertain random variables with different chance distributions

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Abstract. Chance theory is put forward as a tool to deal with a complex system including randomness and uncertainty. As a fundamental concept in chance theory, uncertain random variable is an extension of random variable and uncertain variable. This paper obtains a new law of large numbers for independent uncertain random variables but not necessarily identically distributed.

Keywords: Uncertain variable, uncertain random variable, law of large numbers

1. Introduction

Probability theory has been developing for a long time to describe randomness, a type of indeterminacy phenomena, associated with frequency. However, a fundamental assumption of applying probability theory is that the estimated probability is close enough to the long-run cumulative frequency. To obtain the frequency, we need to do numerous experiments and obtain lots of observed data. Unfortunately, we cannot always gain observed data for economic, technical or some other reasons in practice. At this time, we should rely on domain experts’ belief degrees about the chances that the possible events may happen. However, Kahneman and Tversky [5] said that human tends to overweight unlikely events. Liu [15] showed that human beings usually estimate a much wider range of values than the object actually takes. That is, the conservative estimation of human beings makes the belief degrees deviate from the frequency. Hence, it is not reasonable for us to apply probability theory to modeling belief degrees. Some convincing examples appeared in Liu [13].

Uncertainty theory was founded by Liu [7] to model uncertainty associated with belief degrees. The belief degree is described by uncertain measure which satisfies normality, duality, subadditivity and product axioms. Then Liu [10] perfected uncertainty theory with presenting product uncertain measure which is very different from probability’s. And some basic concepts were presented, such as uncertain variable for modeling uncertain quantity, uncertainty distribution for describing an uncertain variable and inverse uncertainty distribution for convenient calculations. Additionally, Peng and Iwamura [20] gave a sufficient and necessary condition for the uncertainty distribution of an uncertain variable. As an important characteristic to describe uncertain variables, expected value was introduced by Liu [7]. Up to now, uncertainty theory has became an almost complete mathematical system, and lots of researchers have obtained some useful results such as uncertain programming (Liu [9]), uncertain reliability and risk analysis (Liu [12]), uncertain process (Liu [8], Yao and Li [22]), and uncertain differential equation (Liu [8], Gao [2]).

In many cases, both components having samples and components having no samples exist simultaneously in a complex system. That is, this system
contains not only random variables but also uncertain variables. Obviously, we cannot deal with this complex system simply by probability theory or uncertainty theory. In order to describe this complex phenomenon, Liu [16] pioneered chance theory including basic concepts of chance measure, uncertain random variable and chance distribution. To rank the uncertain random variables, Liu [16] presented the concepts of expected value and variance. Since such a theory was established, it has been successfully applied in many fields such as uncertain random programming (Liu [17], Ke, Su and Ni [6]), uncertain random risk analysis (Liu and Ralescu [18]), uncertain random reliability analysis (Wen and Kang [21], Gao and Yao [3]), uncertain random graph and uncertain random network (Liu [14]), uncertain random process (Gao and Yao [1], Yao and Gao [23]), and uncertain random logic (Liu and Yao [19]).

Yao and Gao [24] studied a law of large numbers for independent and identically distributed (iid) uncertain random variables. This paper will consider a law of large numbers for independent uncertain random variables not necessarily identically distributed. The rest of this paper is organized as follows. We will review some basic concepts and properties about uncertainty theory and chance theory in Section 2. We will devote Section 3 to discussing a new law of large numbers for a class of uncertain random variables. Finally, we will make a summary of this paper in Section 4.

2. Preliminaries

In this section, we introduce some fundamental concepts and properties concerning uncertainty theory and chance theory.

2.1. Uncertainty theory

Let \( \Gamma \) be a nonempty set, and \( \mathcal{L} \) a \( \sigma \)-algebra over \( \Gamma \). Each element \( \Lambda \) in \( \mathcal{L} \) is called an event and assigned a number \( \mathcal{M}(\Lambda) \) to indicate the belief degree that we believe \( \Lambda \) will happen. In order to deal with belief degrees rationally in mathematics, Liu [7] suggested the following three axioms:

**Axiom 1. (Normality Axiom)** \( \mathcal{M}(\Gamma) = 1 \) for the universal set \( \Gamma \);

**Axiom 2. (Duality Axiom)** \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any event \( \Lambda \);

**Axiom 3. (Subadditivity Axiom)** For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
\mathcal{M} \left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).
\]

**Definition 2.1. (Liu [7])** The set function \( \mathcal{M} \) is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

The triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space. Furthermore, the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) is defined by Liu [10] as follows:

**Axiom 4. (Product Axiom)** Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying

\[
\mathcal{M} \left( \prod_{k=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Lambda_k)
\]

where \( \Lambda_k \) are arbitrary events chosen from \( \mathcal{L}_k \) for \( k = 1, 2, \ldots \), respectively.

**Definition 2.2. (Liu [7])** An uncertain variable is a measurable function \( \xi \) from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set

\[
\{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \}
\]

is an event.

To describe an uncertain variable, Liu [16] proposed a concept of uncertainty distribution as follows.

**Definition 2.3. (Liu [7])** Suppose \( \xi \) is an uncertain variable. Then the uncertainty distribution of \( \xi \) is defined by

\[
\Phi(x) = \mathcal{M} \{ \xi \leq x \}
\]

for any real number \( x \).

An uncertainty distribution \( \Phi(x) \) is said to be regular if its inverse function \( \Phi^{-1}(\alpha) \) exists and is unique for each \( \alpha \in (0, 1) \). Inverse uncertainty distribution plays an important role in the operation of independent uncertain variables.

The operational law of independent uncertain variables was given by Liu [11] in order to calculate the uncertainty distribution of strictly increasing or
decreasing function. Before introducing the operational law, we review the concept of independence of uncertain variables firstly.

**Definition 2.4.** (Liu [10]) The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_n \) are said to be independent if

\[
M \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \prod_{i=1}^n M \{\xi_i \in B_i\}
\]

for any Borel sets \( B_1, B_2, \ldots, B_n \).

**Theorem 2.1.** (Liu [11]) Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with continuous uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If the function \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_m \) and strictly decreasing with respect to \( x_{m+1}, x_{m+2}, \ldots, x_n \), then the uncertain variable

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]

has an uncertainty distribution

\[
\Phi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \left( \min_{1 \leq i \leq m} \Phi_i(\xi_i) \wedge \min_{m+1 \leq i \leq n} (1 - \Phi_i(\xi_i)) \right).
\]

Moreover, if \( \Phi_1, \Phi_2, \ldots, \Phi_n \) are regular, then \( \xi \) has an inverse uncertainty distribution

\[
\Phi^{-1}(\alpha) = f(\Phi^{-1}_1(\alpha), \ldots, \Phi^{-1}_n(\alpha)),
\]

\[
\Phi^{-1}_1(1 - \alpha), \ldots, \Phi^{-1}_n(1 - \alpha).
\]

To rank an uncertain variable, Liu [16] proposed a concept of expected value as follows.

**Definition 2.5.** (Liu [7]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_0^{+\infty} M[\xi \geq x]dx - \int_{-\infty}^0 M[\xi \leq x]dx
\]

provided that at least one of the two integrals is finite.

**Theorem 2.2.** (Liu [11]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Then

\[
E[\xi] = \int_{-\infty}^{+\infty} xd\Phi(x).
\]

### 2.2 Chance theory

A chance space is virtually a product space \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\), where \((\Gamma, \mathcal{L}, \mathcal{M})\) is an uncertainty space and \((\Omega, \mathcal{A}, \Pr)\) is a probability space. Essentially, it is another triplet,

\[
(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \Pr)
\]

where \(\Gamma \times \Omega\) is the universal set, \(\mathcal{L} \times \mathcal{A}\) is the product \(\sigma\)-algebra, and \(\mathcal{M} \times \Pr\) is the chance measure.

**Definition 2.6.** (Liu [16]) Let \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space, and let \(\Theta \in \mathcal{L} \times \mathcal{A}\) be an event. Then the chance measure of \(\Theta\) is defined as

\[
\text{Ch}(\Theta) = \int_0^1 \Pr[\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x] dx.
\]

Liu [16] proved that the chance measure is normalized, self-dual and increasing, in other words, \(\text{Ch}(\Gamma \times \Omega) = 1\), \(\text{Ch}(\Theta) + \text{Ch}(\Theta^C) = 1\) and \(\text{Ch}(\Theta_1) \leq \text{Ch}(\Theta_2)\) with \(\Theta_1 \subset \Theta_2\). Additionally, Hou [4] verified that the chance measure satisfies subadditivity. That is, for any countable sequence of events \(\Theta_1, \Theta_2, \ldots\), we have

\[
\text{Ch}

\left\{ \bigcup_{i=1}^{\infty} \Theta_i \right\} \leq \sum_{i=1}^{\infty} \text{Ch}(\Theta_i).
\]

**Definition 2.7.** (Liu [16]) An uncertain random variable is a measurable function \(\xi\) from a chance space \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) to the set of real numbers, i.e.,

\[
\{\xi \in B\} = \{(\omega, \gamma) \in \Omega \times \Gamma \mid \xi(\omega, \gamma) \in B\}
\]

is an event for any Borel set \(B\).

To describe an uncertain random variable, Liu [16] proposed a concept of chance distribution as follows.

**Definition 2.8.** (Liu [16]) Suppose \(\xi\) is an uncertain random variable. Then the chance distribution of \(\xi\) is defined by

\[
\Phi(x) = \text{Ch}[\xi \leq x]
\]

for any real number \(x\).

**Definition 2.9.** (Yao and Gao [24]) Suppose \(\Phi(x), \Phi_i(x)\) are chance distributions of uncertain random variables \(\xi, \xi_i\) for \(i = 1, 2, \ldots\), respectively. Then \(\xi\) is said to converge in distribution to \(\xi\) if

\[
\lim_{i \to \infty} \Phi_i(x) = \Phi(x)
\]

for any real number \(x\) at which \(\Phi\) is continuous.
To rank an uncertain random variable, Liu [16] proposed a concept of expected value as follows.

**Definition 2.10.** (Liu [16]) Let $\xi$ be an uncertain random variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_0^{+\infty} \text{Ch}[\xi \geq x]dx - \int_{-\infty}^0 \text{Ch}[\xi \leq x]dx$$

provided that at least one of two integrals is finite.

**Theorem 2.3.** (Liu [16]) Let $\xi$ be an uncertain random variable with chance distribution $\Phi$. Then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx.$$

**Theorem 2.4.** (Yao and Gao [24]) Let $\eta_1, \eta_2, \ldots$ be a sequence of iid random variables with a common probability distribution $\Phi$, and $\tau_1, \tau_2, \ldots$ be a sequence of iid regular uncertain variables. Define

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n)$$

for $n \geq 1$ where $f(x, y)$ is a continuous and strictly monotone function. Then

$$\frac{S_n}{n} \rightarrow \int_{-\infty}^{+\infty} f(x, \tau_1)d\Phi(x)$$

as $n \rightarrow \infty$ in the sense of convergence in distribution.

### 3. Law of large numbers for uncertain random variables

Yao and Gao [24] studied a law of large numbers for identically distributed uncertain random variables. Sometimes the uncertain random variables are not always identically distributed, so we study a law of large numbers for a class of uncertain random variables without different chance distributions.

**Theorem 3.1.** Let $\eta_1, \eta_2, \ldots$ be a sequence of iid random variables with a common probability distribution $\Phi(x)$, and $\tau_1, \tau_2, \ldots$ be a sequence of independent uncertain variables with uncertainty distributions $\Psi_1(x), \Psi_2(x), \ldots$, respectively. Suppose $f(x, y)$ is a continuous and strictly monotone function, we define

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n), \quad n \geq 1.$$

If $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i)d\Phi(x)$ exists in the sense of distribution, then

$$\frac{S_n}{n} \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i)d\Phi(x)$$

in chance distribution as $n \rightarrow \infty$.

**Proof.** According to the definition of convergence in distribution, it is equivalent to prove

$$\lim_{n \rightarrow \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \frac{1}{n} \sum_{i=1}^{n} f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n) \right\}$$

$$\leq \int_{-\infty}^{f(x, y)d\Phi(x)}$$

$$= \text{Ch} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i)d\Phi(x) \right\}$$

$$\leq \int_{-\infty}^{f(x, y)d\Phi(x)}$$

$$= \mathcal{M} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i)d\Phi(x) \right\}$$

$$\leq \int_{-\infty}^{f(x, y)d\Phi(x)}$$

$$\triangleq \Psi(y).$$

We break the proof into two cases according to the monotonicity of the function $f$.

**Case 1:** Suppose $f(x, y)$ is a strictly increasing function with respect to $y$.

Since $\tau_1, \tau_2, \ldots$ are a sequence of independent uncertain random variables and $f(x, y)$ is a continuous function, we know that $f(x, \tau_1), f(x, \tau_2), \ldots$ are a sequence of independent uncertain variables for any $x \in \mathfrak{X}$. Due to $f(x, y)$ being a strictly increasing function with respect to $y$, we obtain the uncertainty distributions of $f(x, \tau_1), f(x, \tau_2), \ldots$ which are

$$\Psi_i(y) = \mathcal{M} \{ f(x, \tau_i) \leq f(x, y) \}$$

$$= \mathcal{M} \left\{ \int_{-\infty}^{f(x, \tau_i)} f(x, \tau_i)d\Phi(x) \right\}$$

$$\leq \int_{-\infty}^{f(x, y)d\Phi(x)}$$

$$\triangleq \Psi(y).$$
where $i = 1, 2, \ldots$. Thus it follows from Theorem 2.1 that

$$\mathcal{M}\left\{ \frac{1}{n} \sum_{i=1}^{n} f(x, \tau_i) \leq \frac{1}{n} \sum_{i=1}^{n} f(x, y) \right\}$$

$$= \sup_{\eta} \min_{\psi(y_i) \Rightarrow \psi(n, y)} \mathcal{M}\left\{ f(x, \tau_i) \leq f(x, y) \right\} \leq n^{1/\psi}$$

Since $\eta_1, \eta_2, \ldots$ are a sequence of iid random variables with a common probability distribution, $f(\eta_1, y)$, $f(\eta_2, y), \ldots$ are also a sequence of iid random variables for any $y \in \mathfrak{F}$. On the one hand, according to the strong law of large numbers for random variables, we have

$$\frac{f(\eta_1, y) + \ldots + f(\eta_n, y)}{n} \to \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \text{ a.s.}$$

for any given number $y$ and any given $\epsilon > 0$. That is, there exists a positive number $N_1$ such that

$$\Pr\left\{ \frac{f(\eta_1, y - \epsilon) + f(\eta_2, y - \epsilon) + \ldots + f(\eta_n, y - \epsilon)}{n} \leq \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \right\}$$

$$\geq \Pr\left\{ \frac{f(\eta_1, y - \epsilon) + f(\eta_2, y - \epsilon) + \ldots + f(\eta_n, y - \epsilon)}{n} \leq \int_{-\infty}^{\infty} f(x, y - \epsilon)d\Phi(x) + \epsilon \right\}$$

$$\geq 1 - \epsilon$$

for any $n \geq N_1$. Thus

$$\text{Ch}\left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \right\}$$

$$= \text{Ch}\left\{ \frac{f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n)}{n} \leq \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \right\}$$

$$= \int_{0}^{1} \Pr\left\{ \mathcal{M}\left\{ \frac{f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n)}{n} \leq \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \right\} \geq r \right\} dr$$

$$\geq \int_{0}^{1} \Pr\left\{ \int_{-\infty}^{\infty} f(x, y)d\Phi(x) \leq \mathcal{M}\left\{ \frac{f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n)}{n} \right\} \right\}$$

$$\geq \int_{0}^{\Psi(y - \epsilon)} \Pr\left\{ \frac{f(\eta_1, y - \epsilon) + \ldots + f(\eta_n, y - \epsilon)}{n} \right\}$$
On the other hand, since \( f(\eta_1, y + \varepsilon), f(\eta_2, y + \varepsilon), \ldots \) are a consequence of \( iid \) random variables, for any given real number \( y \) and any given \( \varepsilon > 0 \), it follows from the strong law of large numbers for random variables that

\[
\frac{f(\eta_1, y + \varepsilon) + f(\eta_2, y + \varepsilon) + \ldots + f(\eta_n, y + \varepsilon)}{n} \rightarrow \int_{-\infty}^{\infty} f(x, y + \varepsilon) \, d\Phi(x), \ a.s.
\]

That is, there exists a positive number \( N_2 \) such that

\[
\Pr\left\{ \frac{f(\eta_1, y + \varepsilon) + f(\eta_2, y + \varepsilon) + \ldots + f(\eta_n, y + \varepsilon)}{n} \leq \int_{-\infty}^{\infty} f(x, y + \varepsilon) \, d\Phi(x) - \varepsilon \right\} \geq 1 - \varepsilon
\]

for any \( n \geq N_2 \). Thus

\[
\text{Ch} \left\{ \frac{S_n}{n} > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y) \, d\Phi(x) \right\} = \text{Ch} \left\{ \frac{f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n)}{n} > \int_{-\infty}^{\infty} f(x, y) \, d\Phi(x) \right\} = 1 - \Psi(y + \varepsilon),
\]

we obtain

\[
\text{Ch} \left\{ \frac{S_n}{n} > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y) \, d\Phi(x) \right\} \geq \text{Pr} \left\{ \frac{f(\eta_1, y + \varepsilon) + \ldots + f(\eta_n, y + \varepsilon)}{n} > \int_{-\infty}^{\infty} f(x, y) \, d\Phi(x) \right\} \geq 1 - \Psi(y + \varepsilon).
\]
for any \( n \geq N_1 \). From Inequality (2), we have
\[
\text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y) \Phi(x) \right\}
\]
\[
\leq 1 - (1 - \epsilon)(1 - \Psi(y + \epsilon))
\]
for any \( n \geq N_1 \) and \( y \in \mathbb{R} \). By Inequalities (1) and (3), we obtain
\[
\text{lim}_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y) \Phi(x) \right\}
\]
\[
= \mathbb{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, \tau) \Phi(x) \right\}
\]
\[
\leq \int_{-\infty}^{\infty} f(x, y) \Phi(x) \right\}
\]
which is equivalent to
\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, y) \Phi(x) \right\}
\]
\[
= \mathbb{M} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x, \tau) \Phi(x) \right\}
\]
\[
\leq \int_{-\infty}^{\infty} f(x, y) \Phi(x) \right\}
\]
The proof is finished. \( \square \)

**Remark 3.1.** For a sequence of iid random variables \( \eta_1, \eta_2, \ldots \) with a common probability distribution \( \Phi(x) \) and \( f(x) \) is a continuous and strictly monotone function, denote \( S_n = f(\eta_1) + f(\eta_2) + \ldots + f(\eta_n) \) for \( n \geq 1 \). Then \( \frac{S_n}{n} \) converges to \( E[f(\eta_1)] \) in distribution. When the limit is a constant convergence in distribution is equivalent to convergence in probability measure, and convergence in probability measure is equivalent to convergence a.s. for the sum of independent random variables. So, if \( E[f(\eta_1)] = c < +\infty \), then \( \frac{S_n}{n} \) converges to \( E[f(\eta_1)] \) a.s., which is just the result in probability theory.

**Remark 3.2.** For a sequence of independent and identically distributed uncertain variables \( \tau_1, \tau_2, \ldots \) and \( f(x) \) is a continuous and strictly monotone function, denote \( S_n = f(\tau_1) + f(\tau_2) + \ldots + f(\tau_n) \) for \( n \geq 1 \). Then \( \frac{S_n}{n} \) converges to \( f(\tau_1) \) in distribution.

**Example 3.1.** For a sequence of iid random variables \( \eta_1, \eta_2, \ldots \) with a common probability distribution \( \Phi(x) \) and a sequence of iid uncertain variables \( \tau_1, \tau_2, \ldots \), denote \( S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \ldots + f(\eta_n, \tau_n) \) for \( n \geq 1 \). Then \( \frac{S_n}{n} \) converges to \( \int_{-\infty}^{\infty} f(x, \tau_1) \Phi(x) \) in distribution. This is the result in Yao and Gao [24].

**Example 3.2.** For a sequence of iid random variables \( \eta_1, \eta_2, \ldots \) with a common probability distribution \( \Phi(x) \), denote \( S_n = \eta_1 + \eta_2 + \ldots + \eta_n \) for \( n \geq 1 \). Then \( \frac{S_n}{n} \) converges to \( E\eta_1 \) in distribution.
Example 3.3. If $\tau_1, \tau_2, \ldots$ are iid, then $\frac{S_n}{n}$ converges to $\tau_1$ in distribution.

4. Conclusion

This paper studied the convergence in distribution for a sequence of functions of random and uncertain variables where random variables are independent and identically distributed and uncertain variables are only independent. The conclusion showed that the average of these uncertain random variables converges in distribution to uncertain variables under some conditions.

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