Stability in inverse distribution for uncertain differential equations

Xiangfeng Yang\textsuperscript{a,b}, Yaodong Ni\textsuperscript{a,*} and Yansheng Zhang\textsuperscript{b}

\textsuperscript{a}School of Information Technology and Management, University of International Business and Economics, Beijing, China
\textsuperscript{b}Department of Mathematical Sciences, Tsinghua University, Beijing, China

Abstract. Uncertain differential equation is a kind of differential equations driven by Liu processes, many concepts of stability for uncertain differential equations have been researched. This paper first presents a concept of convergence in inverse distribution for an uncertain sequence, and then proposes a concept of stability in inverse distribution for an uncertain differential equation. A sufficient condition is proved for an uncertain differential equation being stable in inverse distribution. Moreover, some examples are given to illustrate the reasonability of stability in inverse distribution.

Keywords: Uncertainty theory, uncertain differential equation, stability, inverse distribution

1. Introduction

For a long period of time, probability theory has been applied extensively as an approach to deal with indeterminacy phenomenon. In 1923, Wiener [29] designed a stochastic process with stationary and independent normal random increments, that is called Wiener process. And then, It\'ô [4] created stochastic calculus to deal with the integral and differential of a stochastic process with respect to standard Wiener process in 1940s. Following that, stochastic differential equation driven by Wiener process was proposed and applied to many areas.

However, before applying probability theory in practice, the obtained probability distribution should be close enough to the real frequency via statistics. Otherwise, we have to invite some experts to evaluate their belief degree about the possible events. Through many surveys, Kahneman and Tversky [5] claimed that human beings usually overweight unlikely events. From another point of view, Liu [12] showed that human beings usually estimate a much wider range of values than the object actually takes. If we still consider human degrees of belief as probability distributions, then we may obtain counterintuitive results (Liu [11]). In order to deal with the human belief degree, Liu [6] built an uncertainty theory that is a branch of mathematics based on normality, duality, subadditivity and product axioms. Nowadays, uncertainty theory have been applied to many areas, such as uncertain programming (Liu [9, 10]), uncertain finance (Liu [8], Chen and Gao [2], Liu et al. [15]), uncertain control (Zhu [30]), and uncertain differential game (Yang and Gao [20, 23]).

In order to describe the dynamic relationship of a series of uncertain variables, Liu [7] proposed a concept of uncertain process. As a special uncertain process, Liu process designed by Liu [8] plays a crucial role in uncertain calculus, just like Wiener process in stochastic calculus. It’s a Lipschitz continuous uncertain process with stationary and independent normal uncertain increments. After
that, Liu [8] initiated uncertain calculus to deal with the integral and differential of an uncertain process with respect to Liu process. Uncertain differential equation driven by Liu process was studied by Liu [7]. Later on, Chen and Liu [1] proved the existence and uniqueness theorem for the solution of an uncertain differential equation, and they gave an analytic solution for a linear uncertain differential equation. Liu [14] and Yao [26] provided some methods to solve two types of nonlinear uncertain differential equations. More importantly, Yao and Chen [27] proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations. This work is called Yao-Chen formula that relates an uncertain differential equation and ordinary differential equations. Furthermore, some numerical methods for solving general uncertain differential equations were designed among others by Yao and Chen [27], Yang and Shen [21], Yang and Ralescu [22], Wang et al. [19] and Gao [3]. Recently, Yang and Yao [24] extended uncertain differential equation to uncertain partial differential equation and presented uncertain heat equation.

2. Preliminaries

In this section, we review some basic concepts and important theorems including uncertain variable, uncertain calculus and uncertain differential equation. Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies normality, duality, subadditivity and product axioms. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2.1. (Liu [6]) An uncertain variable $\xi$ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, such that, for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{y \in \Gamma | \xi(y) \in B\}$$

is an event.

In order to describe uncertain variable in practice, the uncertainty distribution $\Phi : \mathcal{M} \to [0, 1]$ of an uncertain variable $\xi$ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. Peng and Iwamura [16] proved that a function $\Phi : \mathcal{M} \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to $x$ at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.$$

If $\xi$ is an uncertain variable with regular uncertainty distribution $\Phi$, then we call the inverse function $\Phi^{-1}(\alpha)$ as the inverse uncertainty distribution of $\xi$.

Example 2.1. An uncertain variable $\xi$ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & x < a \\ (x - a)/(b - a), & a \leq x < b \\ 1, & x \geq b \end{cases}$$

denoted by $\mathcal{L}(a, b)$, where $a$ and $b$ are real numbers with $a < b$. The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Example 2.2. An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(x - \mu)}{\sqrt{3\sigma}}\right)\right)^{-1}, \quad x \in \mathcal{M}$$

denoted by $\mathcal{L}(\mu, \sigma)$. The inverse uncertainty distribution of normal uncertain variable $\mathcal{L}(\mu, \sigma)$ is

$$\Phi^{-1}(\alpha) = \mu - \frac{\sigma}{\pi} \log\left(\frac{1}{1 - \alpha}\right).$$
denoted by \( \mathcal{N}(e, \sigma) \), where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \). The inverse uncertainty distribution of normal uncertain variable \( \mathcal{N}(e, \sigma) \) is
\[
\Phi^{-1}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

**Example 2.3.** An uncertainty variable \( \xi \) is called lognormal if it has a lognormal uncertainty distribution
\[
\Phi(x) = \left(1 + \exp \left( \frac{\pi(e - \ln x)}{\sqrt{3} \sigma} \right) \right)^{-1}, \quad x \geq 0
\]
denoted by \( \mathcal{L}\mathcal{G}\mathcal{N}(e, \sigma) \), where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \). The inverse uncertainty distribution of lognormal uncertain variable \( \mathcal{L}\mathcal{G}\mathcal{N}(e, \sigma) \) is
\[
\Phi^{-1}(\alpha) = \exp \left( e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right).
\]

**Definition 2.2.** (Liu [6]) The uncertain variables \( \xi_1, \xi_2, \cdots, \xi_n \) are said to be independent if
\[
\mathcal{M} \left\{ \bigcap_{i=1}^{n} (\xi_i \in B_i) \right\} = \prod_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}
\]
for any Borel sets \( B_1, B_2, \cdots, B_n \) of real numbers.

**Definition 2.3.** (Liu [6]) Let \( \Phi, \Phi_1, \Phi_2, \cdots \) are the uncertainty distributions of uncertain variables \( \xi, \xi_1, \xi_2, \cdots \), respectively. We say the uncertain sequence \( \{\xi_i\} \) converges in distribution to \( \xi \) if
\[
\lim_{i \to \infty} \Phi_i(x) = \Phi(x)
\]
for all \( x \) at which \( \Phi \) is continuous.

An uncertain process is essentially a sequence of uncertain variables indexed by time. A formal definition of uncertain process is as follows.

**Definition 2.4.** (Liu [7]) Let \( T \) be a totally ordered set (e.g. time) and let \( (\Gamma, \mathcal{L}, \mathcal{M}) \) be an uncertainty space. An uncertain process is a function \( X_t(\gamma) \) from \( T \times (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set of real numbers such that \( \{X_t \in B\} \) is an event for any Borel set \( B \) of real numbers at each time \( t \).

An uncertain process \( X_t \) is said to have independent increments if
\[
X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}
\]
average uncertain variables where \( t_0 \) is the initial time and \( t_1, t_2, \cdots, t_k \) are any times with \( t_0 < t_1 < \cdots < t_k \). An uncertain process \( X_t \) is said to have stationary increments if, for any given \( t > 0 \), the increments \( X_{s+t} - X_s \) are identically distributed uncertain variables for all \( s > 0 \).

**Definition 2.5.** (Liu [8]) An uncertain process \( C_t \) is said to be a Liu process if
(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,
(ii) \( C_t \) has stationary and independent increments,
(iii) every increment \( C_{t+s} - C_s \) is a normal uncertain variable with an uncertainty distribution
\[
\Phi_t(x) = \left(1 + \exp \left( -\frac{\pi x}{\sqrt{3} \sigma} \right) \right)^{-1}, \quad x \in \mathbb{R}.
\]

**Definition 2.6.** (Liu [8]) Let \( X_t \) be an uncertain process and let \( C_t \) be a Liu process. For any partition of closed interval \( [a, b] \) with \( a = t_1 < t_2 < \cdots < t_{k+1} = b \), the mesh is written as
\[
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
\]

Then Liu integral of \( X_t \) with respect to \( C_t \) is
\[
\int_a^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process \( X_t \) is said to be integrable.

**Definition 2.7.** (Liu [7]) Suppose \( C_t \) is a Liu process, and \( f \) and \( g \) are two given functions. Then
\[
dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t
\]
is called an uncertain differential equation. A solution is an uncertain process that satisfies \( (1) \) identically in \( t \).

The existence and uniqueness theorem of solution of uncertain differential equation was proved by Chen and Liu [1] under linear growth condition and Lipschitz continuous condition. More importantly, Yao and Chen [27] proved that the solution of an uncertain differential equation can be represented by a spectrum of ordinary differential equations.

**Definition 2.8.** (Yao and Chen [27]) Let \( \alpha \) be a number with \( 0 < \alpha < 1 \). An uncertain differential equation
\[
dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t
\]
is said to have an \( \alpha \)-path \( X^\alpha_t \) if it solves the corresponding ordinary differential equation
\[
dX^\alpha_t = f(t, X^\alpha_t) \, dt + |g(t, X^\alpha_t)| \Phi^{-1}(\alpha) \, dt
\]
where \( \Phi^{-1}(\alpha) \) is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,
\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.
\]

**Theorem 2.1.** (Yao-Chen Formula [27]) Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t.
\]
Then
\[
\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha,
\]
\[
\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.
\]

3. Convergence in inverse distribution

In this section, we study the convergence in inverse distribution of an uncertain sequence and prove a sufficient and necessary condition for an uncertain sequence converging in inverse distribution.

**Definition 3.1.** Let \( \xi, \xi_1, \xi_2, \cdots \) be an uncertain sequence with regular uncertainty distributions \( \Phi, \Phi_1, \Phi_2, \cdots \), respectively. We say that \( \{\xi_n\} \) converges in inverse distribution to \( \xi \) if
\[
\lim_{n \to \infty} \Phi_n^{-1}(\alpha) = \Phi^{-1}(\alpha)
\]
for all \( \alpha \in (0, 1) \).

**Theorem 3.1.** Let \( \xi, \xi_1, \xi_2, \cdots \) be an uncertain sequence with regular uncertainty distributions \( \Phi, \Phi_1, \Phi_2, \cdots \), respectively. Then \( \{\xi_n\} \) converges in inverse distribution to \( \xi \) if and only if it converges in distribution to \( \xi \).

**Proof.** Firstly, we prove the necessary condition. Assume that
\[
\lim_{n \to \infty} \Phi_n^{-1}(\alpha) = \Phi^{-1}(\alpha)
\]
for all \( \alpha \in (0, 1) \).

1) When \( \Phi(x) = \alpha \in (0, 1) \), for any given \( \epsilon \), we can take \( \epsilon_1 = \Phi^{-1}(\alpha + \epsilon) - \Phi^{-1}(\alpha) > 0 \). There exists \( N_1 \) such that
\[
|\Phi_n^{-1}(\alpha + \epsilon) - \Phi^{-1}(\alpha + \epsilon)| < \epsilon_1 = \Phi^{-1}(\alpha + \epsilon) - \Phi^{-1}(\alpha)
\]
holds for any \( n > N_1 \). Then we have
\[
\Phi_n^{-1}(\alpha + \epsilon) > \Phi^{-1}(\alpha) = x.
\]
That is,
\[
\Phi_n(x) - \Phi(x) < \epsilon.
\]

On the other hand, we take \( \epsilon_2 = \Phi^{-1}(\alpha) - \Phi^{-1}(\alpha - \epsilon) > 0 \). There exists \( N_2 \) such that
\[
|\Phi_n^{-1}(\alpha - \epsilon) - \Phi^{-1}(\alpha - \epsilon)| < \epsilon_2 = \Phi^{-1}(\alpha) - \Phi^{-1}(\alpha - \epsilon)
\]
holds for any \( n > N_2 \). Then we have
\[
\Phi_n^{-1}(\alpha - \epsilon) < \Phi^{-1}(\alpha) = x.
\]
That is,
\[
\Phi_n(x) - \Phi(x) > -\epsilon.
\]

By Equations (2) and (3), taking \( N = \max\{N_1, N_2\} \), we have
\[
|\Phi_n(x) - \Phi(x)| < \epsilon
\]
for any \( n > N \).

2) When \( \Phi(x) = 0 \), for any given \( \epsilon \), we can take \( \epsilon_1 = \Phi^{-1}(\epsilon) - \Phi^{-1}(0) > 0 \). There exists \( N \) such that
\[
|\Phi_n^{-1}(\epsilon) - \Phi^{-1}(\epsilon)| < \epsilon_1 = \Phi^{-1}(\epsilon) - \Phi^{-1}(0)
\]
holds for any \( n > N \). Then we have
\[
\Phi_n^{-1}(\epsilon) > \Phi^{-1}(0) = x.
\]
That is,
\[
0 \leq \Phi_n(x) - \Phi(x) < \epsilon.
\]
When \( \Phi(x) = 1 \), we can also prove this with the same way.

3) When \( \Phi(x) = 1 \), for any given \( \epsilon \), we take \( \epsilon_1 = \Phi^{-1}(1) - \Phi^{-1}(1 - \epsilon) > 0 \). There exists \( N \) such that
\[
|\Phi_n^{-1}(1 - \epsilon) - \Phi^{-1}(1 - \epsilon)| < \epsilon_1 = \Phi^{-1}(1) - \Phi^{-1}(1 - \epsilon)
\]
holds for any \( n > N \). Then we have
\[
\Phi_n^{-1}(1 - \epsilon) < \Phi^{-1}(1) = x.
\]
That is,
\[
0 \geq \Phi_n(x) - \Phi(x) > -\epsilon.
\]

Hence,
\[
\lim_{n \to \infty} \Phi_n(x) = \Phi(x), \quad \forall x \in \mathbb{R}.
\]
That is \( \{\xi_n\} \) converges in distribution to \( \xi \).

Next, we prove the sufficient condition. Assume that
\[
\lim_{n \to \infty} \Phi_n(x) = \Phi(x), \quad \forall x \in \mathbb{R}.
\]

For any \( \alpha \in (0, 1) \) such that \( \Phi^{-1}(\alpha) = x \), we can take \( \epsilon_1 = \Phi(x+\epsilon) - \Phi(x) > 0 \) for any given \( \epsilon > 0 \).

That is, the uncertain sequence \( \{\xi_n\} \) converges in inverse distribution to \( \xi \).

By Theorem 3.1, we have
\[
\lim_{n \to +\infty} \Phi_n(x) = \Phi(x).
\]

Hence, the sequence \( \{\xi_n\} \) converges in distribution to \( \xi \).

**Example 3.2.** Let \( \xi, \xi_1, \xi_2, \ldots \) be normal uncertain variables \( N(e, \sigma) \), \( N(e_1, \sigma_1) \), \( N(e_2, \sigma_2) \), \ldots, respectively. The inverse uncertainty distributions of \( \xi, \xi_1, \xi_2, \ldots \) are
\[
\Phi^{-1}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},
\]

respectively. If \( \lim_{n \to +\infty} e_n = e \), and \( \lim_{n \to +\infty} \sigma_n = \sigma \), then
\[
\lim_{n \to +\infty} \Phi_n^{-1}(\alpha) = e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} = \Phi^{-1}(\alpha).
\]

Hence, the sequence \( \{\xi_n\} \) converges in distribution to \( \xi \).

**Example 3.3.** Let \( \xi, \xi_1, \xi_2, \ldots \) be lognormal uncertain variables \( \mathcal{LOGN}(e, \sigma) \), \( \mathcal{LOGN}(e_1, \sigma_1) \), \( \mathcal{LOGN}(e_2, \sigma_2) \), \ldots, respectively. The inverse uncertainty distributions of \( \xi, \xi_1, \xi_2, \ldots \) are
\[
\Phi^{-1}(\alpha) = \exp \left( e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right),
\]

respectively. If \( \lim_{n \to +\infty} a_n = a \), and \( \lim_{n \to +\infty} b_n = b \), then
\[
\Phi_n^{-1}(\alpha) = \exp \left( e_n + \frac{\sigma_n \sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right),
\]

respectively.
respectively. If \( \lim_{n \to +\infty} e_n = e \), and \( \lim_{n \to +\infty} \sigma_n = \sigma \), then

\[
\lim_{n \to +\infty} \Phi_n^{-1}(\alpha) = \lim_{n \to +\infty} \exp\left( e_n + \frac{\sigma_n \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) = \exp\left( e + \frac{\sigma \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) = \Phi^{-1}(\alpha).
\]

That is, the uncertain sequence \( \{\xi_n\} \) converges in inverse distribution to \( \xi \). By Theorem 3.1, we have

\[
\lim_{n \to +\infty} \Phi_n(x) = \Phi(x).
\]

Hence, the sequence \( \{\xi_n\} \) converges in distribution to \( \xi \).

### 4. Stability in inverse distribution

In this section, we discuss the stability in inverse distribution for uncertain differential equations and prove a sufficient condition for an uncertain differential equation being stable in inverse distribution.

**Definition 4.1.** Let \( X_t \) and \( Y_t \) be two solutions of uncertain differential equation

\[
dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (6)
\]

with different initial values \( X_0 \) and \( Y_0 \), respectively. Assume the inverse uncertainty distribution of \( X_t \) and \( Y_t \) are \( \Phi_t^{-1}(\alpha) \) and \( \Psi_t^{-1}(\alpha) \), respectively. Then the uncertain differential Equation (6) is said to be stable in inverse distribution if for any given \( \epsilon > 0 \), there exists a real number \( \delta \) such that

\[
|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)| < \epsilon
\]

holds for any \( t \geq 0 \), provided \( |X_0 - Y_0| < \delta \).

**Example 4.1.** Consider the linear uncertain differential equation

\[
dX_t = \mu dt + \sigma dC_t. \quad (7)
\]

The two solutions with different initial values \( X_0 \) and \( Y_0 \) are

\[
X_t = X_0 + \mu t + \sigma C_t,
\]

\[
Y_t = Y_0 + \mu t + \sigma C_t
\]

whose inverse uncertainty distributions are

\[
\Phi_t^{-1}(\alpha) = X_0 + \mu t + \sigma \frac{t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},
\]

\[
\Psi_t^{-1}(\alpha) = Y_0 + \mu t + \sigma \frac{t \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},
\]

respectively. Then

\[
\lim_{|X_0 - Y_0| \to 0} |\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)| = 0, \quad \forall t > 0, \alpha \in (0, 1).
\]

Thus, the uncertain differential Equation (7) is stable in inverse distribution.

**Example 4.2.** Consider the linear uncertain differential equation

\[
dX_t = X_t dt + \sigma dC_t. \quad (8)
\]

The two solutions with different initial values \( X_0 \) and \( Y_0 \) are

\[
X_t = \exp(t)X_0 + \sigma \exp(t) \int_0^t \exp(-s) dC_s,
\]

\[
Y_t = \exp(t)Y_0 + \sigma \exp(t) \int_0^t \exp(-s) dC_s
\]

whose the inverse uncertainty distributions are

\[
\Phi_t^{-1}(\alpha) = \exp(t)X_0 + \sigma \exp(t) \frac{(1 - \exp(-t)) \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},
\]

\[
\Psi_t^{-1}(\alpha) = \exp(t)Y_0 + \sigma \exp(t) \frac{(1 - \exp(-t)) \sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha},
\]

respectively. Then

\[
|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)| = \exp(t)|X_0 - Y_0| \to +\infty
\]

as \( t \to +\infty \) Thus, the uncertain differential equation (8) is not stable in inverse distribution.

**Theorem 4.1.** The uncertain differential equation

\[
dY_t = f(t, X_t) dt + g(t, X_t) dC_t
\]

is stable in inverse distribution if the coefficients \( f(t, x) \) and \( g(t, x) \) satisfy the strong Lipschitz condition

\[
|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L(t)|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0
\]
where \( L(t) \) is a positive function satisfying
\[
\int_0^{+\infty} L(t) dt < +\infty.
\]

**Proof.** Assume that \( X_t \) and \( Y_t \) are the solutions of the uncertain differential equation with different initial values \( X_0 \) and \( Y_0 \), respectively. By Theorem 2.1, the inverse uncertainty distributions \( \Phi_t^{-1}(\alpha) \) and \( \Psi_t^{-1}(\alpha) \) of \( X_t \) and \( Y_t \) satisfy the ordinary differential equations
\[
d\Phi_t^{-1}(\alpha) = f(t, \Phi_t^{-1}(\alpha)) dt + |g(t, \Phi_t^{-1}(\alpha))| \Upsilon^{-1}(\alpha) dt,
\]
\[
d\Psi_t^{-1}(\alpha) = f(t, \Psi_t^{-1}(\alpha)) dt + |g(t, \Psi_t^{-1}(\alpha))| \Upsilon^{-1}(\alpha) dt,
\]
respectively, where
\[
\Upsilon^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.
\]

Then for each \( \alpha \in (0, 1) \), we have
\[
|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)|
= \left| \left( X_0 + \int_0^t (f(s, \Phi_s^{-1}(\alpha)) + |g(s, \Phi_s^{-1}(\alpha))| \Upsilon^{-1}(\alpha) ds \right)
- \left( Y_0 + \int_0^t (f(s, \Psi_s^{-1}(\alpha)) + |g(s, \Psi_s^{-1}(\alpha))| \Upsilon^{-1}(\alpha) ds \right) \right|
\leq |X_0 - Y_0| + \int_0^t |f(s, \Phi_s^{-1}(\alpha)) - f(s, \Psi_s^{-1}(\alpha))| ds
+ |\Upsilon^{-1}(\alpha)| \int_0^t |g(s, \Phi_s^{-1}(\alpha)) - g(s, \Psi_s^{-1}(\alpha))| ds
\leq |X_0 - Y_0| + \int_0^t |f(s, \Phi_s^{-1}(\alpha)) - f(s, \Psi_s^{-1}(\alpha))| ds
+ |\Upsilon^{-1}(\alpha)| \int_0^t |g(s, \Phi_s^{-1}(\alpha)) - g(s, \Psi_s^{-1}(\alpha))| ds
\leq |X_0 - Y_0| + \int_0^t L(s) |\Phi_s^{-1}(\alpha) - \Psi_s^{-1}(\alpha)| ds
+ |\Upsilon^{-1}(\alpha)| \int_0^t L(s) |\Phi_s^{-1}(\alpha) - \Psi_s^{-1}(\alpha)| ds
\]
(\text{strong Lipschitz condition})
\[
= |X_0 - Y_0|
+(1 + |\Upsilon^{-1}(\alpha)|) \int_0^t L(s) |\Phi_s^{-1}(\alpha) - \Psi_s^{-1}(\alpha)| ds.
\]

By the Grönwall’s inequality, we get
\[
|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)|
\leq |X_0 - Y_0| \exp \left( (1 + |\Upsilon^{-1}(\alpha)|) \int_0^t L(s) ds \right).
\]

Since
\[
\int_0^{+\infty} L(t) dt < +\infty,
\]
there exists a real number \( N > 0 \) such that
\[
\exp \left( (1 + |\Upsilon^{-1}(\alpha)|) \int_0^t L(s) ds \right) < N
\]
for any \( t \geq 0 \). Thus, for any given \( \epsilon > 0 \), we choose \( \delta = \epsilon/N \) such that
\[
|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)|
\leq |X_0 - Y_0| \exp \left( (1 + |\Upsilon^{-1}(\alpha)|) \int_0^t L(s) ds \right)
< \delta N = \epsilon
\]
for any \( t \geq 0 \) provided \(|X_0 - Y_0| < \delta\). Therefore, the uncertain differential equation is stable in inverse distribution. The theorem is thus proved.

**Example 4.3.** Consider the nonlinear uncertain differential equation
\[
dX_t = \exp(-t) dt + \exp(-t - X_t^2) dC_t. \quad (9)
\]

Note that \( f(t, x) = \exp(-t) \) and \( g(t, x) = \exp(-t - x^2) \) satisfy the strong Lipschitz condition
\[
|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)|
\leq \exp(-t)|x - y|, \quad \forall x, y \in \mathbb{R}, t \geq 0
\]
with
\[
\int_0^{+\infty} \exp(-t) dt < +\infty.
\]

From Theorem 4.1, the uncertain differential Equation (9) is stable in inverse distribution.

In fact, Theorem 4.1 gives a sufficient condition but not a necessary condition for an uncertain differential equation being stable in inverse distribution.

**Example 4.4.** Consider the linear uncertain differential equation
\[
dX_t = -X_t dt + \sigma dC_t, \quad (10)
\]
The two solutions with different initial values $X_0$ and $Y_0$ are

$$X_t = \exp(-t)X_0 + \sigma \exp(-t) \int_0^t \exp(s)\,dC_s,$$

$$Y_t = \exp(-t)Y_0 + \sigma \exp(-t) \int_0^t \exp(s)\,dC_s,$$

whose the inverse uncertainty distributions are

$$\Phi_t^{-1}(\alpha) = \exp(-t)X_0 + \sigma \exp(-t) \frac{(\exp(t) - 1)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},$$

$$\Psi_t^{-1}(\alpha) = \exp(-t)Y_0 + \sigma \exp(-t) \frac{(\exp(t) - 1)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},$$

respectively. Then we have

$$|\Phi_t^{-1}(\alpha) - \Psi_t^{-1}(\alpha)| = \exp(-t)|X_0 - Y_0| \to 0$$

as $t \to +\infty$. Thus, the uncertain differential Equation (10) is stable in inverse distribution. However, the coefficient $f(x, t) = -x$ doesn’t satisfy the strong Lipschitz condition in Theorem 4.1.

**Theorem 4.2.** The linear uncertain differential equation

$$dX_t = (u_{1t}X_t + u_{2t})\,dt + (v_{1t}X_t + v_{2t})\,dC_t$$

(11)

is stable in inverse distribution if $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are real functions such that

$$\int_0^{+\infty} |u_{1t}|\,dt < +\infty \text{ and } \int_0^{+\infty} |v_{1t}|\,dt < +\infty.$$

**Proof.** Note that $f(t, x) = u_{1t}x + u_{2t}$ and $g(t, x) = v_{1t}x + v_{2t}$ in Theorem 4.1. Then

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)|$$

$$= |u_{1t}| |x - y| + |v_{1t}| |x - y|$$

$$= (|u_{1t}| + |v_{1t}|) |x - y|.$$ 

Taking $L(t) = |u_{1t}| + |v_{1t}|$, we have

$$\int_0^{+\infty} L(t)\,dt = \int_0^{+\infty} |u_{1t}|\,dt$$

$$+ \int_0^{+\infty} |v_{1t}|\,dt < +\infty.$$

Thus the linear uncertain differential Equation (11) is stable in inverse distribution. □

5. Conclusion

This paper studied the convergence in inverse distribution of uncertain sequence and then proved a sufficient and necessary condition for an uncertain sequence converging in inverse distribution. Moreover, it also proposed the stability in inverse distribution of uncertain differential equation and gave a sufficient condition for an uncertain differential equation being stable in inverse distribution.

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References


