The p-distance of uncertain variables

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Abstract. In order to research the properties of distance for uncertain variables, we introduced a new concept of p-distance. Then the properties of nonnegativity, identification, symmetry, and triangle inequality for the p-distance of uncertain variables and uncertain vectors were obtained. In the end, we deduced the metric space which is constituted by p-distance is complete.

Keywords: Uncertain variable, uncertain vector, uncertainty theory, p-distance

1. Introduction

In order to describe the subjective uncertain phenomenon, Liu [4] defined an uncertain measure and developed uncertainty theory. A classical application of uncertainty theory was given by Liu [13]. So far, many researchers have contributed in this area.

In uncertainty theory, uncertain variable (Liu [4]) is one of the most important concepts, which is defined as a measurable function from an uncertainty space to the set of real numbers. A sufficient and necessary condition of uncertainty distribution was proved by Peng and Iwamura [15] in 2010. To estimate an uncertainty distribution, uncertain statistic was proposed by Liu [10]. Then Liu [6] introduced uncertain programming which is a type of mathematical programming involving uncertain variables. Liu and Ha [9] proposed a formula to calculate the expected value of functions of uncertain variables. Dai and Chen [2] studied the linearity of the entropy of functions for uncertain variables. Liu [10] presented a measure inversion theorem from which we can calculate the uncertain measures of some events. After introduced the definition of independence by liu [7], Liu [10] presented the operational law of uncertain variables. To model dynamic uncertain phenomena, Liu [5] put forward a concept of uncertain process. After that, Liu [7] designed an important type of uncertain process called canonical process, which is an uncertain process with independent and stationary normal increments. Based on canonical process, uncertain differential was founded by Liu [5] and uncertain calculus was given by Liu [7]. Then Yao [16] proposed uncertain calculus with renewal process in 2012. In addition, uncertain set was proposed by Liu [11] and further researched by Liu [14] in 2012. Up to date, uncertain set has been applied to uncertain inference (Gao [3] and Chen [1]) and uncertain logic (Liu [12] and Li and Liu [8]).

Distance is a powerful tool for pattern recognition and cluster analysis. In order to study pattern recognition and cluster analysis in uncertainty environment, Liu [4] gave the concept of distance based on expected value operator of uncertain sets and uncertain variables, but it does not meet the triangle inequality of distance axioms. The purpose of this paper is to give a new definition of distance of uncertain variables, which is fully satisfying distance axioms.

This paper is devoted to mathematical properties of p-distance of uncertain variables. The rest of this paper is organized as follows. Uncertain space and some basic contents and theorems of uncertainty theory will be introduced in Section 2. Then the definition and properties of p-distance of uncertain variables will be proposed in Section 3. After that, some definitions and theorems of uncertain vectors
will be investigated in Section 4. At last, a brief summary will be given.

2. Preliminary

In this section, we recalled some definitions and theorems in uncertainty theory, which will be used in this paper.

Let $\Gamma$ be a nonempty set, and let $\mathcal{L}$ be a $\sigma$–algebra over $\Gamma$. A number $\mathcal{M}(\Lambda)$ indicates the level that each element $\Lambda \in \mathcal{L}$ (which is called an event) will occur. Liu [4] proposed the set function $\mathcal{M}$, which is called uncertain measures if it satisfies the following three axioms:

**Axiom 1.** (Normality) $\mathcal{M}(\Gamma) = 1$.

**Axiom 2.** (Monotonicity) $\mathcal{M}(\Lambda_1) \leq \mathcal{M}(\Lambda_2)$ whenever $\Lambda_1 \subset \Lambda_2$.

**Axiom 3.** (Self-Duality) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$, for any event $\Lambda$.

Next, the definition of uncertain space is introduced.

**Definition 2.1.** (Liu [4]) Let $\Gamma$ be a nonempty set, let $\mathcal{L}$ be a $\sigma$–algebra over $\Gamma$, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

The property of uncertain measure is recalled in the following theorem.

**Theorem 2.1.** (Countable Subadditivity, Liu [4]) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).$$

**Definition 2.2.** (Liu [4]) An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set $B$ of real numbers.

**Definition 2.3.** (Liu [4]) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{-\infty}^{+\infty} \mathcal{M}(\xi \geq x) dx - \int_{-\infty}^{0} \mathcal{M}(\xi \leq x) dx,$$

provided that at least one of the two integrals is finite.

**Definition 2.4.** (Liu [4]) The distance between uncertain variables $\xi$ and $\eta$ is defined as

$$d(\xi, \eta) = E[|\xi - \eta|].$$

That is, the distance between $\xi$ and $\eta$ is just the expected value of $|\xi - \eta|$. Since $|\xi - \eta|$ is a nonnegative uncertain variable, we always have

$$\tilde{d}(\xi, \eta) = \int_{0}^{+\infty} \mathcal{M}(|\xi - \eta| \geq x) dx.$$

**Theorem 2.2.** (Liu [4]) Let $\xi, \eta, \tau$ be uncertain variables, and let $d(\cdot, \cdot)$ be the distance. Then we have

(a) (Nonnegativity) $\tilde{d}(\xi, \eta) \geq 0$;

(b) (Identification) $\tilde{d}(\xi, \eta) = 0$ if and only if $\xi = \eta$;

(c) (Symmetry) $\tilde{d}(\xi, \eta) = \tilde{d}(\eta, \xi)$;

(d) (Pseudo Triangle Inequality) $\tilde{d}(\xi, \eta) \leq 2\tilde{d}(\xi, \tau) + 2\tilde{d}(\tau, \eta)$.

**Definition 2.5.** (Liu [4]) A k-dimensional uncertain vector is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that $\{\xi \in B\}$ is an event for any Borel set $B$ of k-dimensional real vectors.

**Theorem 2.3.** (Liu [4]) The vector $(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables.

3. The p-distance of uncertain variables

Based on uncertain measure, a concept of p-distance of uncertain variables is given and some properties are investigated in this section.

**Definition 3.1.** The p-distance between uncertain variables $\xi$ and $\eta$ is defined as

$$d_p(\xi, \eta) = (E[|\xi - \eta|^p])^{1/p}, \quad p > 0.$$

**Example 3.1.** Take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = 0.7, \mathcal{M}\{\gamma_2\} = 0.3$. Define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} 3, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}, \quad \eta(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2. \end{cases}$$

Then

$$|\xi - \eta|^p(\gamma) = \begin{cases} 2^p, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

Thus we have

$$E[|\xi - \eta|^p] = \int_{0}^{1} 1 dx + \int_{1}^{2} 0.7 dx = 1 + 0.7(2^p - 1).$$

Therefore,
\[ d_p(\xi, \eta) = (E[|\xi - \eta|^p])^{\frac{1}{p+1}} = (1 + 0.7(2^p - 1))^{\frac{1}{p+1}}. \]

**Remark 3.1.** The 1-distance between uncertain variables \( \xi \) and \( \eta \) is called simple distance between \( \xi \) and \( \eta \), which is \( d(\xi, \eta) = (E[|\xi - \eta|])^{\frac{1}{2}} \).

Next, we study the properties of \( p \)-distance.

**Theorem 3.1.** Let \( \xi, \eta, \tau \) be uncertain variables, and let \( d_p(\cdot, \cdot) \) be the \( p \)-distance. Then we have

(a) (Nonnegativity) \( d_p(\xi, \eta) \geq 0 \);
(b) (Identification) \( d_p(\xi, \eta) = 0 \) if and only if \( \xi = \eta \);
(c) (Symmetry) \( d_p(\xi, \eta) = d_p(\eta, \xi) \);
(d) (Triangle Inequality) \( d_p(\xi, \eta) \leq d_p(\xi, \tau) + d_p(\tau, \eta) \).

**Proof.** The parts (a) and (c) follow immediately from Definition 3.1. Now we prove the parts (b) and (d).

(b) If \( \xi = \eta \), then \( d_p(\xi, \eta) = 0 \).

If \( \xi \neq \eta \), there exists a point \( \gamma_1 \) such that \( M\{\gamma_1\} > 0 \), \( \xi(\gamma_1) \neq \eta(\gamma_1) \). Then

\[
\begin{align*}
\alpha^p & = \int_0^{+\infty} M\{|\xi - \eta|^p \geq x\} \, dx \\
& \geq M\{\gamma_1\}|\xi(\gamma_1) - \eta(\gamma_1)|^p > 0.
\end{align*}
\]

Thus, \( d_p(\xi, \eta) > 0 \). That is to say, \( d_p(\xi, \eta) = 0 \) means \( \xi = \eta \).

(d) If at least one of \( d_p(\xi, \tau) \), \( d_p(\tau, \eta) \) is zero, then the inequality \( d_p(\xi, \eta) \leq d_p(\xi, \tau) + d_p(\tau, \eta) \) holds.

Let \( d_p(\xi, \tau) > 0 \) and \( d_p(\tau, \eta) > 0 \). For any \( 0 < \alpha < 1 \), it follows from Axiom 2 and Theorem 2.1 that

\[
\begin{align*}
d_p^{p+1}(\xi, \eta) &= \int_0^{+\infty} M\{|\xi - \eta|^p \geq x\} \, dx \\
& = \int_0^{+\infty} M\{|\xi - \eta| \geq x^\frac{1}{p}\} \, dx \\
& \leq \int_0^{+\infty} M\{|\xi - \tau| + |\tau - \eta| \geq x^\frac{1}{p}\} \, dx \\
& \leq \int_0^{+\infty} M\{|\xi - \tau| \geq \alpha x^\frac{1}{p}\} \cup \{|\tau - \eta| \geq (1 - \alpha)x^\frac{1}{p}\} \, dx \\
& \leq \int_0^{+\infty} M\{|\xi - \tau| \geq \alpha x^\frac{1}{p}\} \, dx \\
& + \int_0^{+\infty} M\{|\tau - \eta| \geq (1 - \alpha)x^\frac{1}{p}\} \, dx \\
& = \int_0^{+\infty} M\{|\xi - \tau|^p \geq \alpha^p x\} \, dx \\
& + \int_0^{+\infty} M\{|\tau - \eta|^p \geq (1 - \alpha)^p x\} \, dx \\
& = \frac{E[|\xi - \tau|^p]}{\alpha^p} + \frac{E[|\tau - \eta|^p]}{(1 - \alpha)^p}.
\end{align*}
\]

Especially, if we set

\[
\alpha = \frac{d_p(\xi, \tau)}{d_p(\xi, \tau) + d_p(\tau, \eta)}, \quad 1 - \alpha = \frac{d_p(\tau, \eta)}{d_p(\xi, \tau) + d_p(\tau, \eta)},
\]

then

\[
d_p^{p+1}(\xi, \eta) \leq (d_p(\xi, \tau) + d_p(\tau, \eta))^{p+1}.
\]

That is

\[
d_p(\xi, \eta) \leq d_p(\xi, \tau) + d_p(\tau, \eta).
\]

**Remark 3.2.** Let \( \xi, \eta \) be uncertain variables. If \( \xi \neq \eta \), there exists a point \( \gamma_1 \) such that \( M\{\gamma_1\} > 0 \), \( \xi(\gamma_1) \neq \eta(\gamma_1) \). Then \( E[|\xi - \eta|^p] \geq M\{\gamma_1\}|\xi(\gamma_1) - \eta(\gamma_1)|^p \).

**Example 3.2.** Let \( \xi_1 = (1, 2, 3) \), \( \xi_2 = (6, 7, 8) \), \( \xi_3 = (10, 11, 12) \) be zigzag uncertain variables. Now we consider the relationships among the \( p \)-distance of \( \xi_1, \xi_2, \xi_3 \).

If we take \( p = 1 \), then

\[
\begin{align*}
d(\xi_1, \xi_2) &= (E[|\xi_1 - \xi_2|^1])^{\frac{1}{2}} \approx 2.23, \\
d(\xi_1, \xi_3) &= (E[|\xi_1 - \xi_3|^1])^{\frac{1}{2}} \approx 3.00, \\
d(\xi_2, \xi_3) &= (E[|\xi_2 - \xi_3|^1])^{\frac{1}{2}} \approx 2.00.
\end{align*}
\]

If we take \( p = 2 \), then

\[
\begin{align*}
d_2(\xi_1, \xi_2) &= (E[|\xi_1 - \xi_2|^2])^{\frac{1}{2}} \approx 2.97, \\
d_2(\xi_1, \xi_3) &= (E[|\xi_1 - \xi_3|^2])^{\frac{1}{2}} \approx 4.35, \\
d_2(\xi_2, \xi_3) &= (E[|\xi_2 - \xi_3|^2])^{\frac{1}{2}} \approx 2.58.
\end{align*}
\]

Taking the distance \( \tilde{d}(\cdot, \cdot) \) into consideration, we can get

\[
\tilde{d}(\xi_1, \xi_2) = (E[|\xi_1 - \xi_2|]) \approx 4.99,
\]
\[ \tilde{d}(\xi_1, \xi_3) = (E[|\xi_1 - \xi_3|]) \approx 8.99, \]
\[ \tilde{d}(\xi_2, \xi_3) = (E[|\xi_2 - \xi_3|]) \approx 3.99. \]

It is easy to verify that the p-distance \( d(\cdot, \cdot), d_2(\cdot, \cdot) \) satisfy triangle inequality for zigzag uncertain variables \( \xi_1 = (1, 2, 3), \xi_2 = (6, 7, 8), \xi_3 = (10, 11, 12). \) For distance \( d(\cdot, \cdot), \) we have \( \tilde{d}(\xi_1, \xi_3) > d(\xi_1, \xi_2) + d(\xi_2, \xi_3), \) therefore it does not meet triangle inequality.

**Remark 3.3.** According to Theorem 2.2 and Example 3.2, we know that the distance in Liu [4] does not satisfy triangle inequality, which only meets nonnegativity, identification and symmetry. The p-distance in this paper not only inherits nonnegativity, identification and symmetry of the distance axioms, but also meets triangle inequality. That is, the p-distance completely satisfy the distance axioms, therefore it ought to seek higher theoretical value.

By the definition of p-distance, we deduce the following metric space and investigate its properties.

**Definition 3.2.** Let \( \mathcal{F} \) be the set of uncertain variables which have finite expected values. Then the set \( \mathcal{F} \) with p-distance \( d_p \) is called a metric space of uncertain variables, and is denoted by \( (\mathcal{F}, d_p) \).

Some operational laws for p-distance are given in metric space.

**Theorem 3.2.** Let \( (\mathcal{F}, d_p) \) be metric space. For any uncertain variables \( \xi, \eta, \tau \in \mathcal{F}, \) and \( \lambda \in \mathcal{R}, \) we can get the following two conclusions:

1. \( d_p(\xi + \tau, \eta + \tau) = d_p(\xi, \eta); \)
2. \( d_p(\lambda \xi, \lambda \eta) = |\lambda|^p d_p(\xi, \eta). \)

**Proof.** (1) According to the definition of p-distance, we get

\[ d_p(\xi + \tau, \eta + \tau) = (E[(\xi + \tau) - (\eta + \tau)])^{1/p} = (E[|\xi - \eta|])^{1/p} = d_p(\xi, \eta). \]

(2) It follows from the definition of p-distance that

\[ d_p(\lambda \xi, \lambda \eta) = (E[|\lambda \xi - \lambda \eta|])^{1/p} = (E[|\lambda|^{p}|\xi - \eta|])^{1/p} = |\lambda|^{p} (E[|\xi - \eta|])^{1/p} = |\lambda|^{p} d_p(\xi, \eta). \]

**Definition 3.3.** Suppose that \( \xi, \xi_1, \xi_2, \cdots \) are uncertain variables defined on metric space \( (\mathcal{F}, d_p) \). Then uncertain variables sequence \( \{\xi_i\} \) is said to be convergent in p-distance to \( \xi \) if \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0. \)

**Example 3.3.** Suppose that \( \{\xi_i(\gamma)\} \) is uncertain variables sequence defined on metric space \( (\mathcal{F}, d_p) \). For any positive integer \( i \), there is an integer \( j \) such that \( i = 2^j + k \), where \( k \) is an integer between 0 and \( 2^j - 1 \). The uncertain variables sequence \( \{\xi_i(\gamma)\} \) is defined by

\[ \xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \leq \gamma \leq (k + 1)/2^j \\ 0, & \text{otherwise} \end{cases} \]

for \( i = 1, 2, \cdots, \) and \( \xi \equiv 0. \) Then

\[ \lim_{i \to \infty} E[|\xi_i - \xi|^p] = \lim_{i \to \infty} \int_{k/2^j}^{(k+1)/2^j} 1^p \, dx = \lim_{i \to \infty} \frac{1}{2^j} = 0, \]

therefore,

\[ \lim_{i \to \infty} d_p(\xi_i, \xi) = \lim_{i \to \infty} (E[|\xi_i - \xi|^p])^{1/p} = 0. \]

That is, uncertain variables sequence \( \{\xi_i\} \) converges in p-distance to \( \xi. \)

**Definition 3.4.** Let \( \xi_1, \xi_2, \cdots \) be uncertain variables. Then we call uncertain variables sequence \{\xi_i\} p-Cauchy variables sequence if \( \lim_{i,j \to \infty} d_p(\xi_i, \xi_j) = 0. \)

**Theorem 3.3.** Let \( \xi, \xi_1, \xi_2, \cdots \) be uncertain variables. If \( \lim_{n \to \infty} d_p(\xi_n, \xi) = 0, \) then uncertain variables sequence \( \{\xi_n\} \) is p-Cauchy variables sequence.

**Proof.** Since

\[ \lim_{i,j \to \infty} (d_p(\xi_i, \xi) + d_p(\xi_j, \xi)) = \lim_{i \to \infty} d_p(\xi_i, \xi) + \lim_{j \to \infty} d_p(\xi_j, \xi) = 0, \]

for any \( i, j. \) By Triangle Inequality, we have

\[ \lim_{i,j \to \infty} d_p(\xi_i, \xi_j) \leq \lim_{i \to \infty} d_p(\xi_i, \xi) + \lim_{j \to \infty} d_p(\xi_j, \xi) = 0. \]

Therefore,

\[ \lim_{i,j \to \infty} d_p(\xi_i, \xi_j) = 0. \]

It follows from Definition 3.4 that uncertain variables sequence \( \{\xi_n\} \) is p-Cauchy variables sequence.
Theorem 3.4. If uncertain variables sequence \( \{ \xi_i \} \) is \( p \)-Cauchy variables sequence, then there is an uncertain variable \( \xi \) in metric space \( (\mathcal{F}, d_p) \) such that \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \). That is, metric space \( (\mathcal{F}, d_p) \) is complete.

Proof. Suppose that \( \Gamma^+ \) is the set of points which have positive uncertain measure. If \( \{ \xi_i \} \) is \( p \)-Cauchy sequence, for any \( \gamma \in \Gamma^+ \),

\[
\lim_{i, j \to \infty} |\xi_i(\gamma) - \xi_j(\gamma)|^p \leq \lim_{i, j \to \infty} \frac{E[|\xi_i(\gamma) - \xi_j(\gamma)|^p]}{\mathcal{M}\{\gamma\}} = \lim_{i, j \to \infty} \frac{d_p^{p+1}(\xi_i(\gamma), \xi_j(\gamma))}{\mathcal{M}\{\gamma\}} = 0.
\]

Thus, \( \{ \xi_i(\gamma) \} \) is a Cauchy sequence of real numbers. Let \( \xi(\gamma) \) be the number such that \( \lim_{i \to \infty} \xi_i(\gamma) = \xi(\gamma) \), for any \( \gamma \in \Gamma^+ \). It is clear that \( \xi \) is an uncertain variable.

For any \( k > 0 \), there exists integer \( i_k > 0 \) such that \( E[|\xi_i - \xi|^p] \leq \frac{1}{4^k} \), \( i, j \geq i_k \). According to Axiom 2, we know that

\[
E[|\xi_i - \xi|^p] = E[\sum_{l \geq k} (\xi_i - \xi_{i+1})^p] = \int_0^{+\infty} \mathcal{M}\{\sum_{l \geq k} (\xi_i - \xi_{i+1})^p \geq x\}dx
\]

\[
\leq \int_0^{+\infty} \mathcal{M}\{\bigcup_{l \geq k} \{\xi_i - \xi_{i+1} \geq \frac{x}{2^l}\}\}dx
\]

\[
\leq \sum_{l \geq k} \mathcal{M}\{|\xi_i - \xi_{i+1}| \geq \frac{x}{2^l}\}dx
\]

\[
= \sum_{l \geq k} 2^l \int_0^{+\infty} \mathcal{M}\{|\xi_i - \xi_{i+1}|^p \geq x\}dx
\]

\[
= \sum_{l \geq k} 2^p E[|\xi_i - \xi_{i+1}|^p] \leq \sum_{l \geq k} \frac{2^p}{2^{l-1}} = \frac{2^p}{2(k-1)}.
\]

That is, \( \lim_{k \to \infty} d_p(\xi_i_k, \xi) = \lim_{k \to \infty} (E[|\xi_i - \xi|^p])^{\frac{1}{p+1}} = 0 \).

Let \( i_k, i \to \infty \), we have

\[
d_p(\xi_i, \xi) \leq d_p(\xi_i, \xi_i_k) + d_p(\xi_i_k, \xi) \to 0.
\]

The proof is complete.

Theorem 3.5. Let \( \xi, \eta \) be uncertain variables defined on metric space \( (\mathcal{F}, d_p) \), and let \( \{ \xi_i \}, \{ \eta_i \} \) be uncertain variables sequences defined on metric space \( (\mathcal{F}, d_p) \). Then we obtain the following conclusions:

(a) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then \( \lim_{i \to \infty} d_p(|\xi_i|, |\xi|) = 0 \).

(b) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then \( \lim_{i \to \infty} d_p(\xi_i, \eta) = 0 \), then \( \xi = \eta \).

(c) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then \( \lim_{i \to \infty} d_p(\eta_i, \eta) = 0 \), then \( \lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta) = 0 \), for any real numbers \( a, b \).

(d) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then \( \lim_{i \to \infty} d_p(\eta_i, \eta) = 0 \) and there exists a real number \( M > 0 \) such that \( |\xi| \leq M, |\eta| \leq M \), then \( \lim_{i \to \infty} d_p(\xi_i \eta_i, \xi \eta) = 0 \).

Proof. (a) It follows from Definition 3.1 that

\[
0 = \lim_{i \to \infty} d_p(\xi_i, \xi) = \lim_{i \to \infty} E[|\xi_i - \xi|^p]^{\frac{1}{p+1}} = \lim_{i \to \infty} d_p(|\xi_i|, |\xi|).
\]

Therefore,

\[
\lim_{i \to \infty} d_p(|\xi_i|, |\xi|) = 0.
\]

(b) According to Triangle Inequality, we have

\[
0 \leq \lim_{i \to \infty} d_p(\xi, \eta) \leq \lim_{i \to \infty} d_p(\xi_i, \xi) + \lim_{i \to \infty} d_p(\xi_i, \eta) = 0,
\]

thus, \( \xi = \eta \).

(c) By Triangle Inequality, we can get

\[
0 \leq \lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta) \leq \lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta)
\]

\[
+ \lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta) = 0,
\]

therefore,

\[
\lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta) = 0.
\]
(d) Suppose that \( \Gamma^+ \) is the set of points which have positive uncertain measure. Then for any \( \gamma \in \Gamma^+ \),
\[
\lim_{i \to \infty} |\xi_i(\gamma) - \xi(\gamma)| \leq \lim_{i \to \infty} \frac{E[|\xi_i - \xi|^p]}{\mathcal{M}(\gamma)} = \lim_{i \to \infty} d_p^{p+1}(\xi_i, \xi) = 0.
\]

Example 4.1. Take \((\xi_1, \xi_2, \cdots, \xi_n)\), \(\eta = (\eta_1, \eta_2, \cdots, \eta_n)\) be uncertain vectors. Let \(\mathcal{M}(\gamma) = 0.7, \mathcal{M}(\gamma_2) = 0.3\). Define four uncertain variables as follows,
\[
\gamma_1(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2, \end{cases}
\]
\[
\gamma_2(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases}
\]
\[
\gamma_1(\gamma) = \begin{cases} 2, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}
\]

4. The \(p\)-distance of uncertain vectors

In this section, we will introduce the definition and properties of \(p\)-distance of uncertain vectors.

Definition 4.1. Let \(\xi = (\xi_1, \xi_2, \cdots, \xi_n)\), \(\eta = (\eta_1, \eta_2, \cdots, \eta_n)\) be uncertain vectors. Then the \(p\)-distance between \(\xi\) and \(\eta\) is defined as
\[
d_p(\xi, \eta) = \left( \sum_{i=1}^{n} d_p^p(\xi_i, \eta_i) \right)^{\frac{1}{p}}.
\]

Example 4.1. Take \((\Gamma, \mathcal{L}, \mathcal{M})\) to be \(\gamma_1, \gamma_2\) with \(\mathcal{M}(\gamma_1) = 0.7, \mathcal{M}(\gamma_2) = 0.3\). Define four uncertain variables as follows,
\[
\gamma_2(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2, \end{cases}
\]
\[
\eta_1(\gamma) = \begin{cases} 2, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}
\]

Let \(\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2)\) be uncertain vectors. Then the \(p\)-distance between \(\xi\) and \(\eta\) is
\[
d_p(\xi, \eta) = (d_p^2(\xi_1, \eta_1) + d_p^2(\xi_2, \eta_2))^\frac{1}{2}
\]
\[
= ((\int_0^1 1 dx)^{\frac{2}{p+1}} + (\int_0^1 0.3 dx)^{\frac{2}{p+1}})^{\frac{1}{2}}
\]
\[
= (1 + 0.3)^{\frac{2}{p+1}}.
\]

If we take \(p = 1\), then \(d(\xi, \eta) = 1.3^{\frac{1}{2}}\).

Theorem 4.1. Let \(\xi, \eta, \tau\) be uncertain vectors, and let \(d_p(\cdot, \cdot)\) be the \(p\)-distance. Then we have
(a) (Nonnegativity) \(d_p(\xi, \eta) \geq 0\);
(b) (Identification) \(d_p(\xi, \eta) = 0\) if and only if \(\xi = \eta\);
(c) (Symmetry) \(d_p(\xi, \eta) = d_p(\eta, \xi)\);
(d) (Triangle Inequality) \(d_p(\xi, \eta) \leq d_p(\xi, \tau) + d_p(\tau, \eta)\).

Proof. The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d).

Let \(\xi = (\xi_1, \xi_2, \cdots, \xi_n), \eta = (\eta_1, \eta_2, \cdots, \eta_n), \tau = (\tau_1, \tau_2, \cdots, \tau_n)\). It follows from Triangle Inequality and Cauchy-Schwarz inequality that
\[
\sum_{i=1}^{n} d_p^2(\xi_i, \eta_i) \leq \sum_{i=1}^{n} (d_p(\xi_i, \tau_i) + d_p(\tau_i, \eta_i))^2
\]
\[
= \sum_{i=1}^{n} (d_p^2(\xi_i, \tau_i) + 2d_p(\xi_i, \tau_i)d_p(\tau_i, \eta_i) + d_p^2(\tau_i, \eta_i))
\]
\[
\leq \sum_{i=1}^{n} d_p^2(\xi_i, \tau_i) + \sum_{i=1}^{n} d_p^2(\tau_i, \eta_i)
\]
\[
+ 2\left( \sum_{i=1}^{n} d_p^2(\xi_i, \tau_i) \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} d_p^2(\tau_i, \eta_i) \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_{i=1}^{n} d_p^2(\xi_i, \tau_i) \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} d_p^2(\tau_i, \eta_i) \right)^{\frac{1}{2}}.
\]

Therefore,
\[
\sum_{i=1}^{n} d_p^2(\xi_i, \eta_i) \leq (\sum_{i=1}^{n} d_p^2(\xi_i, \tau_i))^{\frac{1}{2}} + (\sum_{i=1}^{n} d_p^2(\tau_i, \eta_i))^{\frac{1}{2}}.
\]
That is,
\[ d_p(\xi, \eta) \leq d_p(\xi, \tau) + d_p(\tau, \eta). \]

**Definition 4.2.** Let \( \mathcal{F}^n \) be the set of \( n \)-dimensional uncertain vectors, and let \( d_p \) be p-distance of uncertain vectors. Then \( \mathcal{F}^n \) with p-distance \( d_p \) is called a metric space, and is denoted by \((\mathcal{F}^n, d_p)\).

**Definition 4.3.** Suppose that \( \xi, \xi_1, \xi_2, \cdots \) are \( n \)-dimensional uncertain vectors defined on metric space \( (\mathcal{F}^n, d_p) \). Then uncertain vectors sequence \( \{\xi_i\} \) is said to be convergent in p-distance to \( \xi \) if \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \).

**Definition 4.4.** Let \( \xi, \xi_1, \xi_2, \cdots \) be uncertain vectors. Then we call uncertain vectors sequence \( \{\xi_i\} \) p-Cauchy vectors sequence if \( \lim_{i,j \to \infty} d_p(\xi_i, \xi_j) = 0 \).

**Theorem 4.2.** Let \( \xi, \xi_1, \xi_2, \cdots \) be uncertain vectors. If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then uncertain vectors sequence \( \{\xi_i\} \) is p-Cauchy vectors sequence.

**Proof.** It follows immediately from Definition 4.4 and Triangle Inequality.

**Theorem 4.3.** If uncertain vectors sequence \( \{\xi_i\} \) is p-Cauchy vectors sequence, then there is an uncertain vector \( \xi \) in metric space \( (\mathcal{F}^n, d_p) \) such that \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \). That is, metric space \( (\mathcal{F}^n, d_p) \) is complete.

**Proof.** For any \( i \), we have \( \xi_i = (\xi_{i1}, \xi_{i2}, \cdots, \xi_{in}) \). It can be easily proved that \( \lim_{i,j \to \infty} d_p(\xi_i, \xi_j) = 0 \) if and only if \( \lim_{i,j \to \infty} d_p(\xi_{ik}, \xi_{jk}) = 0 \), for any \( k, 1 \leq k \leq n \).

Then according to Definition 4.1, in a similar proof of Theorem 3.4, we know metric space \( (\mathcal{F}^n, d_p) \) is complete.

**Theorem 4.4.** Let \( \xi, \eta \) be \( n \)-dimensional uncertain vectors defined on metric space \( (\mathcal{F}^n, d_p) \), and let \( \{\xi_i\}, \{\eta_i\} \) be uncertain vectors sequences defined on metric space \( (\mathcal{F}^n, d_p) \). Then we obtain the following conclusions:

(a) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), then \( \lim_{i \to \infty} d_p(\|\xi_i\|, \|\xi\|) = 0 \).

(b) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), \( \lim_{i \to \infty} d_p(\xi_i, \eta) = 0 \), then \( \xi = \eta \).

(c) If \( \lim_{i \to \infty} d_p(\xi_i, \xi) = 0 \), \( \lim_{i \to \infty} d_p(\eta_i, \eta) = 0 \), then \( \lim_{i \to \infty} d_p(a\xi_i + b\eta_i, a\xi + b\eta) = 0 \), for any real numbers \( a, b \).

**Proof.** By Definition 4.1 and Theorem 3.5, it is easy to prove parts (a), (b), and (c).

5. Conclusions

In the setting of uncertainty theory, the concept of p-distance for uncertain variables was proposed. Then the properties of p-distance of uncertain variables were investigated. Based on the definition of p-Cauchy variables sequence, we deduced that metric space formed by uncertain variables and p-distance is complete. Finally, we extended the properties of p-distance of uncertain variables to the case of uncertain vectors.

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