Value-at-risk in uncertain random risk analysis

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A B S T R A C T

Uncertain random variables provide a tool to deal with phenomena in which uncertainty and randomness simultaneously exist. This paper proposes a concept of value-at-risk to quantify the risk of an uncertain random system. In addition, a value-at-risk theorem is proved in order to calculate the value-at-risk, and is applied to series systems, parallel system, k-out-of-n system, standby system, and structural system.

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1. Introduction

In real life we constantly have to make decisions in random environment. To deal with this problem, probabilistic risk analysis was presented by Roy [18] with his risk index (safety-first criterion) which is the probability measure that some specified loss occurs. After that, probabilistic value-at-risk (VaR) was introduced by the leading bank J.P. Morgan [14] as a methodology to evaluate the maximum possible loss when the right tail distribution is ignored. In order to account for not only the probability of loss but also the severity of the loss, Rockafeller and Uryasev [19] presented the probabilistic tail value-at-risk (TVaR) that is the expected outcome conditional on the loss exceeding the VaR of the distribution. Probabilistic risk analysis has been successfully applied in engineering, finance, management science and so on.

In order to obtain the probability distribution function, adequate historical data are required. However, in many cases, there are no samples available to estimate the probability distribution. Thus some domain experts are invited to evaluate the degree of belief that each event may happen. On the one hand, human beings usually overweight unlikely events (Kahneman and Tversky [2]). On the other hand, human beings usually estimate a much wider range of values than the object actually takes (Liu [10]). This conservatism of human beings makes the degree of belief deviate far from the frequency. In order to rationally deal with degree of belief, uncertainty theory was founded by Liu [4] in 2007, and studied by many scholars subsequently. Based on uncertainty theory, uncertain programming was proposed by Liu [6] and was applied to machine scheduling problem, vehicle routing problem and project scheduling problem by Liu [10]. In addition, Liu [7] used uncertainty theory to evaluate the reliability index for uncertain systems. As an initial research on risk analysis in uncertain environment, Liu [7] defined risk as the accidental loss plus uncertain measure of such loss, and presented a risk index that is the uncertain measure that some specific loss occurs. In this way, Liu [7] built a framework of uncertain risk analysis based on his uncertainty theory. In addition, Peng [16] developed an uncertain value-at-risk (VaR) methodology and extended it to a tail value-at-risk (TVaR) methodology.

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This paper will introduce a tool of value-at-risk (VaR) to quantify the risk of an uncertain random system. Then a series of results will be proved in order to calculate the VaR. We will also apply the uncertain random VaR analysis to series systems, parallel systems, k-out-of-n systems, standby systems, and structural system.

2. Value-at-risk

Risk index (Liu and Ralescu [13]) is an important tool to deal with uncertain random system. Sometimes, we need to know how large is the scale of loss once the uncertain random loss happens with some degree. Taking this point of view, we introduce the following VaR metric of loss function, which is intuitively a combination of the chance measure of loss and the scale of loss.

**Definition 1.** Assume that a system contains uncertain random factors $\xi_1, \xi_2, \ldots, \xi_n$, and has a loss function $f$. Then the value-at-risk (VaR) is

$$\text{VaR}(\alpha) = \sup \{x \mid \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) \geq x\} \geq \alpha\}$$

for each given confidence level $\alpha \in (0,1]$. 

When the uncertain random variables degenerate to random variables, the value-at-risk becomes the one in J.P. Morgan [14]. When the uncertain random variables degenerate to uncertain variables, the value-at-risk becomes the one in Peng [16].

**Theorem 1.** The value-at-risk $\text{VaR}(\alpha)$ is decreasing with respect to $\alpha$. That is, if $\alpha_1 < \alpha_2$, then $\text{VaR}(\alpha_1) \geq \text{VaR}(\alpha_2)$.

**Proof.** It is easy to see from the definition of VaR that, if $\alpha_1 < \alpha_2$, then

$$\text{VaR}(\alpha_1) = \sup \{x \mid \text{Ch}\{f(\xi_1, \ldots, \xi_n) \geq x\} \geq \alpha_1\}$$

$$\geq \sup \{x \mid \text{Ch}\{f(\xi_1, \ldots, \xi_n) \geq x\} \geq \alpha_2\} = \text{VaR}(\alpha_2).$$

Thus $\text{VaR}(\alpha)$ is decreasing with respect to $\alpha$. □

**Theorem 2.** Assume that a system contains uncertain random factors $\xi_1, \xi_2, \ldots, \xi_n$, and has a loss function $f$. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ has a continuous chance distribution $\Phi(x)$, then for each confidence level $\alpha \in (0,1]$, we have

$$\text{VaR}(\alpha) = \sup \{x \mid \Phi(x) \leq 1 - \alpha\}.$$

**Proof.** It follows from the chance inversion theorem, duality of chance measure and continuity of chance distribution $\Phi(x)$ that

$$\text{VaR}(\alpha) = \sup \{x \mid \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) \geq x\} \geq \alpha\}$$

$$= \sup \{x \mid 1 - \text{Ch}\{f(\xi_1, \xi_2, \ldots, \xi_n) < x\} \geq \alpha\}$$

$$= \sup \{x \mid 1 - \Phi(x) \geq \alpha\}$$

$$= \sup \{x \mid \Phi(x) \leq 1 - \alpha\}.$$

□

**Theorem 3.** Assume that a system contains uncertain random factors $\xi_1, \xi_2, \ldots, \xi_n$, and has a loss function $f$. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ has a regular chance distribution $\Phi(x)$, then for each confidence level $\alpha \in (0,1]$, we have

$$\text{VaR}(\alpha) = \Phi^{-1}(1 - \alpha).$$
Proof. It follows from Theorem 2 that
\[ \text{VaR}(\alpha) = \sup \{ x | \Phi(x) \leq 1 - \alpha \} \]
\[ = \sup \{ x | x \leq \Phi^{-1}(1 - \alpha) \} \]
\[ = \Phi^{-1}(1 - \alpha). \]
\[ \square \]

Theorem 4. Assume that a system contains independent random variables \( \eta_1, \eta_2, \ldots, \eta_m \) with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \) and independent uncertain variables \( \tau_1, \tau_2, \ldots, \tau_n \) with regular uncertainty distributions \( Y_1, Y_2, \ldots, Y_n \), respectively. If the loss function \( f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \) is strictly increasing with respect to \( \tau_1, \ldots, \tau_k \) and strictly decreasing with respect to \( \tau_{k+1}, \ldots, \tau_n \), then
\[ \text{VaR}(\alpha) = \sup \left\{ x | \int_{y_m} F(x; y_1, \ldots, y_m) d\Psi_1(y_1) d\cdots d\Psi_m(y_m) \leq 1 - \alpha \right\} \]
where \( F(x; y_1, y_2, \ldots, y_m) \) is the root \( \alpha \) of the equation
\[ f(y_1, y_2, \ldots, y_m, Y_1^{-1}(\alpha), \ldots, Y_k^{-1}(\alpha), Y_{k+1}^{-1}(1 - \alpha), \ldots, Y_n^{-1}(1 - \alpha)) = x. \]

Proof. It follows from the operational law of uncertain random variables that \( f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \) has a chance distribution
\[ \Phi(x) = \int_{y_m} F(x; y_1, \ldots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \]
where \( F(x; y_1, y_2, \ldots, y_m) \) is the root \( \alpha \) of the equation
\[ f(y_1, y_2, \ldots, y_m, Y_1^{-1}(\alpha), \ldots, Y_k^{-1}(\alpha), Y_{k+1}^{-1}(1 - \alpha), \ldots, Y_n^{-1}(1 - \alpha)) = x. \]
Applying Theorem 2, we get
\[ \text{VaR}(\alpha) = \sup \left\{ x | \int_{y_m} F(x; y_1, \ldots, y_m) d\Psi_1(y_1) d\cdots d\Psi_m(y_m) \leq 1 - \alpha \right\}. \]
\[ \square \]

3. Series system

Theorem 5. Consider a series system in which there are \( m \) elements whose lifetimes are independent random variables \( \eta_1, \eta_2, \ldots, \eta_m \) with continuous probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \) and \( n \) elements whose lifetimes are independent uncertain variables \( \tau_1, \tau_2, \ldots, \tau_n \) with continuous uncertainty distributions \( Y_1, Y_2, \ldots, Y_n \), respectively. If the loss is understood in case the system fails before time \( T \), then
\[ \text{VaR}(\alpha) = \sup \{ x | (1 - Y_1(T - x)) \land \cdots \land Y_n(T - x) \}
\]
\[ (1 - \Psi_1(T - x)) \land \cdots \land (1 - \Psi_m(T - x)) \leq 1 - \alpha \}. \]

Proof. At first, the lifetime of the series system is \( \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n \). Thus the loss function is
\[ f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) = T - \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n. \]
The chance distribution of \( f \) is
\[ \Phi(x) = \text{Ch}\{T - \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n \leq x\}
\]
\[ = \text{Ch}\{\eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n \geq T - x\}
\]
\[ = \int_{y_m} \text{Ch}\{\eta_1 \land \cdots \land \eta_m \land \tau_1 \land \cdots \land \tau_n \geq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[ = \int_{y_1 \land \cdots \land y_m \geq T - x} \text{Ch}\{\eta_1 \land \cdots \land \eta_m \land \tau_1 \land \cdots \land \tau_n \geq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[ + \int_{y_1 \land \cdots \land y_m \leq T - x} \text{Ch}\{\eta_1 \land \cdots \land \eta_m \land \tau_1 \land \cdots \land \tau_n \leq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[ = 0 + \int_{y_1 \land \cdots \land y_m \leq T - x} M\{\eta_1 \land \cdots \land \eta_m \land \tau_1 \land \cdots \land \tau_n \leq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[ = \int_{y_1 \land \cdots \land y_m \leq T - x} (1 - M\{\tau_1 \land \cdots \land \tau_n \land T - x\}) d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[ = (1 - Y_1(T - x)) \land \cdots \land Y_n(T - x) (1 - \Psi_1(T - x)) \land \cdots \land (1 - \Psi_m(T - x)). \]
By using Theorem 2, the VaR is
\[
\text{VaR}(\alpha) = \sup\{x \mid (1 - Y_1(T - x) \vee \cdots \vee Y_n(T - x)) \\
(1 - \Psi_1(T - x)) \cdots (1 - \Psi_m(T - x)) \leq 1 - \alpha\}.
\]
The proof is completed. \(\square\)

4. Parallel system

Theorem 6. Consider a parallel system in which there are \(m\) elements whose lifetimes are independent random variables \(\eta_1, \eta_2, \ldots, \eta_m\) with continuous probability distributions \(\Psi_1, \Psi_2, \ldots, \Psi_m\) and \(n\) elements whose lifetimes are independent uncertain variables \(\tau_1, \tau_2, \ldots, \tau_n\) with continuous uncertainty distributions \(Y_1, Y_2, \ldots, Y_n\), respectively. If the loss is understood in case the system fails before time \(T\), then
\[
\text{VaR}(\alpha) = \sup\{x \mid (\bigvee_{i=1}^{n} Y_i(T - x)) \Psi_1(T - x) \cdots \Psi_m(T - x) \geq \alpha\}.
\]

Proof. At first, the lifetime of the parallel system is \(\eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n\). Thus the loss function is
\[
f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) = T - \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n.
\]
The chance distribution of \(f\) is
\[
\Phi(x) = \text{Ch}(T - \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \leq x) \\
= \int_{y_1 \vee \cdots \vee y_m \leq T - x} M\{y_1 \vee \cdots \vee y_m \vee \tau_1 \vee \cdots \vee \tau_n \leq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[
+ \int_{y_1 \vee \cdots \vee y_m \geq T - x} M\{y_1 \vee \cdots \vee y_m \vee \tau_1 \vee \cdots \vee \tau_n \geq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[
= \int_{y_1 \vee \cdots \vee y_m \leq T - x} M\{\tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \leq T - x\} d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[
+ \int_{y_1 \vee \cdots \vee y_m \geq T - x} 1 d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
\[
= (1 - Y_1(T - x) \vee \cdots \vee Y_n(T - x)) \Psi_1(T - x) \cdots \Psi_m(T - x) \\
+ 1 - \Psi_1(T - x) \cdots \Psi_m(T - x)
\]
\[
= 1 - (\bigvee_{i=1}^{n}(Y_i(T - x))) \Psi_1(T - x) \cdots \Psi_m(T - x).
\]
By using Theorem 2, the VaR is
\[
\text{VaR}(\alpha) = \sup\{x \mid 1 - (\bigvee_{i=1}^{n} Y_i(T - x)) \Psi_1(T - x) \cdots \Psi_m(T - x) \leq 1 - \alpha\} \\
= \sup\{x \mid (\bigvee_{i=1}^{n} Y_i(T - x)) \Psi_1(T - x) \cdots \Psi_m(T - x) \geq \alpha\}.
\]
The proof is completed. \(\square\)

5. k-Out-of-(m + n) system

Theorem 7. Consider a k-out-of-(m + n) system in which there are \(m\) elements whose lifetimes are independent random variables \(\eta_1, \eta_2, \ldots, \eta_m\) with probability distributions \(\Psi_1, \Psi_2, \ldots, \Psi_m\) and \(n\) elements whose lifetimes are independent uncertain variables \(\tau_1, \tau_2, \ldots, \tau_n\) with regular uncertainty distributions \(Y_1, Y_2, \ldots, Y_n\), respectively. If the loss is understood in case the system fails before time \(T\), then
\[
\text{VaR}(\alpha) = \sup\left\{x \left| \int_{\eta_1, \ldots, \eta_m} F(x; y_1, \ldots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \leq 1 - \alpha \right\}
\]
where \(F(x; y_1, y_2, \ldots, y_m)\) is the root \(\alpha\) of the equation
\[
k\text{-max}\{y_1, y_2, \ldots, y_m, Y_1^{-1}(1 - \alpha), Y_2^{-1}(1 - \alpha), \ldots, Y_n^{-1}(1 - \alpha)\} = T - x.
\]

Proof. At first, the lifetime of the k-out-of-(m + n) system is
\[
k\text{-max}\{\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n\}.
\]
Thus the loss function is
\[
f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) = T - k\text{-max}\{\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n\}.
\]
It follows from the operational law of uncertain random variables that the chance distribution of \( f \) is
\[
\Phi(x) = \text{Ch}\{T - k\text{-max}\{\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n\} \leq x\}
\]
\[
= \text{Ch}\{k\text{-max}\{\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n\} \geq T - x\}
\]
\[
= \int_{y_1, \cdots, y_n} F(x; y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
where \( F(x; y_1, y_2, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
k\text{-max}\{y_1, y_2, \cdots, y_m, \tau_1^{-1}(1 - \alpha), \tau_2^{-1}(1 - \alpha), \cdots, \tau_n^{-1}(1 - \alpha)\} = T - x.
\]

By using Theorem 2, the VaR is
\[
\text{VaR}(\alpha) = \sup \{ x \mid \int_{y_1, \cdots, y_m} F(x; y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \leq 1 - \alpha \}.
\]

\[ \square \]

\section{6. Standby system}

\textbf{Theorem 8.} Consider a standby system in which there are \( m \) elements whose lifetimes are independent random variables \( \eta_1, \eta_2, \cdots, \eta_m \) with probability distributions \( \Psi_1, \Psi_2, \cdots, \Psi_m \) and \( n \) elements whose lifetimes are independent uncertain variables \( \tau_1, \tau_2, \cdots, \tau_n \) with regular uncertainty distributions \( Y_1, Y_2, \cdots, Y_n \), respectively. If the loss is understood in case the system fails before time \( T \), then
\[
\text{VaR}(\alpha) = \sup \{ x \mid \int_{y_1, \cdots, y_m} F(x; y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \leq 1 - \alpha \}
\]
where \( F(x; y_1, y_2, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
\gamma_1^{-1}(1 - \alpha) + \gamma_2^{-1}(1 - \alpha) + \cdots + \gamma_n^{-1}(1 - \alpha) = T - (y_1 + y_2 + \cdots + y_m + x).
\]

\textbf{Proof.} At first, the lifetime of the standby system is \( \eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n \). Thus the loss function is
\[
f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) = T - (\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n).
\]
It follows from the operational law of uncertain random variables that the chance distribution of \( f \) is
\[
\Phi(x) = \text{Ch}\{T - (\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n) \leq x\}
\]
\[
= \text{Ch}\{\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n \geq T - x\}
\]
\[
= \int_{y_1, \cdots, y_m} F(x; y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)
\]
where \( F(x; y_1, y_2, \cdots, y_m) \) is the root \( \alpha \) of the equation
\[
\gamma_1^{-1}(1 - \alpha) + \gamma_2^{-1}(1 - \alpha) + \cdots + \gamma_n^{-1}(1 - \alpha) = T - (y_1 + y_2 + \cdots + y_m + x).
\]
It follows from Theorem 2 that
\[
\text{VaR}(\alpha) = \sup \{ x \mid \int_{y_1, \cdots, y_m} F(x; y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m) \leq 1 - \alpha \}.
\]
The proof is completed. \[ \square \]

\section{7. Structural system}

\textbf{Theorem 9.} Consider a structural system that has \( n \) rods in series and an object. Assume that the strength variables of the \( n \) rods are independent uncertain variables \( \tau_1, \tau_2, \cdots, \tau_n \) with continuous uncertainty distributions \( Y_1, Y_2, \cdots, Y_n \), respectively. We also assume that the gravity of the object is a random variable \( \eta \) with probability distribution \( \Psi \). If the structural system fails whenever the load variable \( \eta \) exceeds at least one of the strength variables \( \tau_1, \tau_2, \cdots, \tau_n \), then
\[
\text{VaR}(\alpha) = \sup \left\{ x \mid \int_{y=0}^{+\infty} \gamma_1(y - x) \vee \cdots \vee \gamma_n(y - x) d\Psi(y) \geq \alpha \right\}.
\]

\textbf{Proof.} At first, the strength of the structural system is \( \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \). Thus the loss function is
\[
f(\eta, \tau_1, \tau_2, \cdots, \tau_n) = \eta - \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n.
\]
The chance distribution of \( f \) is
\[
\Phi(x) = \text{Ch}\{\eta - \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \leq x\}
\]
\[ \begin{align*}
= \int_0^{+\infty} \mathcal{M}(y) \big( y \land \tau_1 \land \tau_2 \land \cdots \land \tau_n \leq x \big) dy \\
= \int_0^{+\infty} \mathcal{M}(y) \big( y \lor \tau_1 \lor \tau_2 \lor \cdots \lor \tau_n \geq y - x \big) dy \\
= \int_0^{+\infty} (1 - \gamma_1(y - x) \lor \cdots \lor \gamma_n(y - x)) dy
\end{align*} \]

By using Theorem 2, we obtain
\[ \text{VaR}(\alpha) = \sup \left\{ x \big| 1 - \int_0^{+\infty} \gamma_1(y - x) \lor \cdots \lor \gamma_n(y - x) dy \leq 1 - \alpha \right\} \]
\[ \text{VaR}(\alpha) = \sup \left\{ x \big| \int_{\mathbb{R}^n} \gamma(y \land y_2 \land \cdots \land y_n + x) dy_1(y_1) \cdots dy_n(y_n) \leq 1 - \alpha \right\}. \]

**Theorem 10.** Consider a structural system that has \( n \) rods in series and an object. Assume that the strength variables of the \( n \) rods are independent random variables \( \eta_1, \eta_2, \ldots, \eta_n \) with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_n \), respectively. We also assume that the gravity of the object is an uncertain variable \( \tau \) with continuous uncertainty distribution \( Y \). If the structural system fails whenever the load variable \( \tau \) exceeds at least one of the strength variables \( \eta_1, \eta_2, \ldots, \eta_n \), then
\[ \text{VaR}(\alpha) = \sup \left\{ x \big| \int_{\mathbb{R}^n} \gamma(y \land y_2 \land \cdots \land y_n + x) dy_1(y_1) \cdots dy_n(y_n) \leq 1 - \alpha \right\}. \]

**Proof.** At first, the strength of the structural system is \( \eta_1 \land \eta_2 \land \cdots \land \eta_n \). Thus the loss function is
\[ f(\eta_1, \eta_2, \ldots, \eta_n, \tau) = \tau - \eta_1 \land \eta_2 \land \cdots \land \eta_n. \]

The chance distribution of  \( f \) is
\[ \Phi(x) = \mathcal{C} \{ \tau \leq \eta_1 \land \eta_2 \land \cdots \land \eta_n \leq x \} \]
\[ = \int_{\mathbb{R}^n} \mathcal{M}(\tau \land y_1 \land y_2 \land \cdots \land y_n \leq x) dy_1(y_1) \cdots dy_n(y_n) \\
= \int_{\mathbb{R}^n} \mathcal{M}(\tau \leq y_1 \land y_2 \land \cdots \land y_n + x) dy_1(y_1) \cdots dy_n(y_n) \\
= \int_{\mathbb{R}^n} \gamma(y \land y_2 \land \cdots \land y_n + x) dy_1(y_1) \cdots dy_n(y_n). \]

By using Theorem 2, we obtain
\[ \text{VaR}(\alpha) = \sup \left\{ x \big| \int_{\mathbb{R}^n} \gamma(y \land y_2 \land \cdots \land y_n + x) dy_1(y_1) \cdots dy_n(y_n) \leq 1 - \alpha \right\}. \]

The proof is completed. \( \square \)

8. Conclusion

This paper introduced a tool of VaR to quantify the risk of an uncertain random system. Then a series of results were proved in order to calculate the VaR. This paper also applied the uncertain random VaR analysis to series systems, parallel systems, \( k \)-out-of-\( n \) system, standby systems, and structural system. In the future, we will apply VAR technique to deal with real uncertain random systems and develop some efficient algorithm to calculate the numerical solutions.

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Appendix A. Uncertain variable

Let \( L \) be a \( \sigma \)-algebra over a nonempty set \( \Gamma \). A set function \( M \) from \( L \) to \([0, 1]\) is called an uncertain measure if it satisfies the following axioms (Liu [4]):

**Axiom 1** (Normality Axiom). \( M(\Gamma) = 1 \) for the universal set \( \Gamma \);

**Axiom 2** (Duality Axiom). \( M(\Lambda) + M(\Lambda^c) = 1 \) for any event \( \Lambda \);

**Axiom 3** (Subadditivity Axiom). For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
M\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).
\]

The triplet \((\Gamma, L, M)\) is said to be an uncertainty space. To deal with compound events, Liu [5] proposed the fourth axiom of uncertainty theory:

**Axiom 4** (Product Axiom). Let \((\Gamma_k, L_k, M_k)\) be uncertainty spaces for \( k = 1, 2, \ldots \). The product uncertain measure \( M \) is an uncertain measure satisfying

\[
M\left( \prod_{k=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} M_k(\Lambda_k)
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( L_k \) for \( k = 1, 2, \ldots \), respectively. An uncertain variable (Liu [4]) is a measurable function \( \xi \) from an uncertainty space \((\Gamma, L, M)\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set \( \{ \xi \in B \} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B \} \) is an event. In order to describe an uncertain variable in practice, the uncertainty distribution was defined by Liu [4] as the following function,

\[
\Phi(x) = M[\xi \leq x], \quad \forall x \in \mathbb{R}.
\]

Peng and Iwamura [15] proved that a function \( \Phi; \mathbb{R} \rightarrow [0, 1] \) is an uncertainty distribution if and only if it is a monotone increasing function except \( \Phi(x) \equiv 0 \) and \( \Phi(x) \equiv 1 \).

**Theorem 11** (Liu [8]). Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If

\[
\begin{align*}
&f(\xi_1, \xi_2, \ldots, \xi_n) \\
&\text{is a strictly increasing function with respect to } \xi_1, \ldots, \xi_m \text{ and strictly decreasing function with respect to } \xi_{m+1}, \ldots, \xi_n.
\end{align*}
\]

then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]

has an inverse uncertainty distribution

\[
\Psi^{-1}(\alpha) = f(\Psi_1^{-1}(\alpha), \ldots, \Psi_m^{-1}(\alpha), \Psi_{m+1}^{-1}(1-\alpha), \ldots, \Psi_n^{-1}(1-\alpha)).
\]

Appendix B. Uncertain random variable

Let \((\Gamma, L, M)\) be an uncertainty space and let \((\Omega, A, \Pr)\) be a probability space. Then the product \((\Gamma, L, M) \times (\Omega, A, \Pr)\) is called a chance space.

**Definition 2** (Liu [11]). Let \((\Gamma, L, M) \times (\Omega, A, \Pr)\) be a chance space, and let \( \Theta \in L \times A \) be an event. Then the chance measure of \( \Theta \) is defined as

\[
\text{Ch}[\Theta] = \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\} \, dr.
\]

**Definition 3** (Liu [11]). An uncertain random variable is a measurable function \( \xi \) from a chance space \((\Gamma, L, M) \times (\Omega, A, \Pr)\) to the set of real numbers, i.e., \( \{ \xi \in B \} \) is an event for any Borel set \( B \).

For any Borel set \( B \), it is clear that the chance measure of the uncertain random event \( \{ \xi \in B \} \) is

\[
\text{Ch}[\xi \in B] = \int_0^1 \Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid \xi(\gamma, \omega) \in B\} \geq r\} \, dr.
\]

It has been proved that the chance measure \( \text{Ch}[\xi \in B] \) is a monotone increasing function of \( B \) and

\[
\text{Ch}[\xi \in \emptyset] = 0, \quad \text{Ch}[\xi \in \Omega] = 1.
\]

In addition, for any Borel set \( B \), we have

\[
\text{Ch}[\xi \in B] + \text{Ch}[\xi \in B^c] = 1.
\]
A chance distribution of an uncertain random variable $\xi$ is defined by $\Phi(x) = \text{Ch}[\xi \leq x]$ for every $x \in \mathbb{R}$. It is known that a function $\Phi : \mathbb{R} \to [0, 1]$ is a chance distribution if and only if it is a monotone increasing function except $\Phi(x) = 0$ and $\Phi(x) = 1$. If the chance distribution $\Phi$ is a continuous function, then for any real number $x$, we have

$$\text{Ch}[\xi \leq x] = \Phi(x), \quad \text{Ch}[\xi \geq x] = 1 - \Phi(x).$$

(7)

The above result is also called the chance inversion theorem.

**Theorem 12** ([Liu [12]]). Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $Y_1, Y_2, \cdots, Y_n$, respectively. If the function

$$f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$$

is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the uncertain random variable

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\eta}^x f(y_1, \cdots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)$$

(9)

where $F(x; y_1, \cdots, y_m)$ is the root $\alpha$ of the equation

$$f(y_1, \cdots, y_m, Y_1^{-1}(\alpha), \cdots, Y_k^{-1}(\alpha), Y_{k+1}^{-1}(1 - \alpha), \cdots, Y_n^{-1}(1 - \alpha)) = x.$$

**References**


