Stability in $p$-th moment for uncertain differential equation with jumps

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Abstract. Uncertain differential equation with jumps is a type of differential equations driven by both Liu process and the uncertain renewal process. Stability of uncertain differential equation with jumps, playing an important role in uncertain differential equation with jumps, means insensitivity of the state of a system with a small changes in the initial state. This paper focuses on the stability in $p$-th moment for uncertain differential equation with jumps, including the concept of stability in $p$-th moment, and the sufficient condition for uncertain differential equation with jumps being stable in $p$-th moment. In addition, the relationship between stability in measure and stability in $p$-th moment for the uncertain differential equation with jumps is also discussed.

Keywords: Uncertainty theory, uncertain differential equation, stability, uncertain process

1. Introduction

For handling dynamic stochastic systems, Ito proposed stochastic differential equation, a type of differential equation driven by Wiener process. Following that, stochastic differential equation has been widely researched by many scholars. Kalman and Bucy [12] put forward a method to filter noise by using the stochastic differential equation. And Black and Scholes [1] obtained the famous European option pricing formulas for a stock under the assumption that the price of a stock followed stochastic differential equation.

Except for random phenomena in reality, human’s uncertainty associated with belief degree is another different type of indeterminate phenomenon. For dealing with human’s uncertainty, Liu [1] presented uncertainty theory in 2007 and refined it in 2009 [2]. For modelling the evolution of uncertain phenomena, the uncertain process was proposed by Liu [3]. Then in 2009, Liu [2] designed Liu process which is a Lipschitz continuous uncertain process with stationary and independent normal uncertain increments. Meanwhile, Liu [2] founded uncertain calculus to handle the integral and differential of an uncertain process with respect to Liu process. There are many other references, such an Yang and Gao [4], Chen and Ralescu [5], and Gao and Chen [6].

differential equation and ordinary differential equations was called Yao-Chen formula. After that, Yao [13] showed the extreme value and integral of the solution of an uncertain differential equation can also be represented by the solutions of the ordinary differential equations. Meanwhile, some numerical methods for solving uncertain differential equations were given by Yang and Ralescu [14], Yao and Chen [12], and Gao [15]. Stability of a differential equation means that a perturbation on the initial value will not result in an influential shift. So far, there are mainly six types of stability for an uncertain differential equation, namely stability in measure, stability in mean, stability in $p$-th moment, stability in inverse distribution, stability in exponent, and almost sure stability. The concept of stability in measure for an uncertain differential equation was first presented by Liu [2], and Yao et al. [16] gave a sufficient condition for an uncertain differential equation being stable in measure. After that, Yao et al. [17] discussed the stability in mean for uncertain differential equation. Sheng and Wang [18] considered the stability in $p$-th moment for uncertain differential equation. Liu et al. [19] investigated the almost sure stability for uncertain differential equation. Yao et al. [20] introduced the exponential stability for uncertain differential equation. Yang et al. [21] studied the stability in inverse distribution for an uncertain differential equation and proved a sufficient condition for an uncertain differential equation being stable in inverse distribution.

Uncertain differential equation with jumps, a type of differential equations driven by both Liu process and the uncertain renewal process, was first proposed by Yao [25] in 2012. Yao [26] gave the solutions of two special types of uncertain differential equations with jumps, he gave a sufficient condition for an uncertain differential equation with jumps having a unique solution, and he studied the stability in measure for an uncertain differential equation with jumps. Following that, Ji and Ke [27] discussed almost sure stability for an uncertain differential equation with jumps.

Extending the previous work on stability of uncertain differential equation with jumps, this paper will develop a concept of stability in $p$-th moment for uncertain differential equation with jumps and give a sufficient condition for uncertain differential equation with jumps having stability in $p$-th moment. The relationship between the stability in $p$-th moment and the stability in measure for uncertain differential equation with jumps is also discussed. The rest of the paper is organized as follows. In Section 2, we review some basic concepts in uncertain theory and uncertain differential equation with jumps. In Section 3, we propose the stability in $p$-th moment and prove a sufficient condition for uncertain differential equation with jumps having stability in $p$-th moment. In Section 4, the relationship between the stability in measure and the stability in $p$-th moment for uncertain differential equation with jumps is discussed. At last, some conclusions are given in Section 5.

2. Preliminaries

In this part, we review some preliminary concepts and theorems in uncertainty theory and uncertain differential equation with jumps.

Definition 2.1. (Liu [1]) Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set $\Gamma$.

Axiom 2. (Duality Axiom) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event $\Lambda$.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, Liu [2] defined a product uncertain measure by the fourth axiom:

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Lambda_k)$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.2. (Liu [1]) An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$, the set $\{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event in $\mathcal{L}$.
Definition 2.3. (Liu [1]) The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by

\[
\Phi(x) = M[\xi \leq x]
\]

for any real number \( x \).

Peng and Iwamura [23] proved that a function \( \Phi : \mathbb{R} \to [0, 1] \) is an uncertainty distribution if and only if it is a monotone increasing function except \( \Phi(x) \equiv 0 \) and \( \Phi(x) \equiv 1 \). An uncertainty distribution \( \Phi(x) \) is said to be regular if it is a continuous and strictly increasing function with respect to \( x \) at which \( 0 < \Phi(x) < 1 \), and

\[
\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.
\]

Definition 2.4. (Liu [22]) Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi(x) \). Then the inverse function \( \Phi^{-1}(\alpha) \) is called the inverse uncertainty distribution of \( \xi \).

Example 2.1. An uncertainty variable \( \xi \) is called linear if it has a linear uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < a, \\
(x-a)/(b-a), & \text{if } a \leq x < b, \\
1, & \text{if } x \geq b
\end{cases}
\]

denoted by \( \mathcal{L}(a, b) \), where \( a \) and \( b \) are real numbers with \( a < b \). The inverse uncertainty distribution of linear uncertain variable \( \mathcal{L}(a, b) \) is

\[
\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b.
\]

Example 2.2. An uncertainty variable \( \xi \) is called normal if it has a normal uncertainty distribution

\[
\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3\sigma}}\right)\right)^{-1}, \quad x \in \mathbb{R}
\]

denoted by \( \mathcal{N}(e, \sigma) \), where \( e \) and \( \sigma \) are real numbers with \( \sigma > 0 \). The inverse uncertainty distribution of normal uncertain variable \( \mathcal{N}(e, \sigma) \) is

\[
\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.
\]

Definition 2.5. (Liu [1]) Let \( \xi \) be an uncertain variable on an uncertainty space \( (\Gamma, \mathcal{L}, M) \). Then its expected value \( E(\xi) \) is

\[
E[\xi] = \int_{-\infty}^{+\infty} M[\xi \geq x]dx - \int_{-\infty}^{0} M[\xi \leq x]dx
\]

provided that at least one of the two integrals is finite.

Theorem 2.1. (Liu [1]) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). Then

\[
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx.
\]

An uncertain process is a sequence of uncertain variables indexed by time. A formal definition of uncertain process is stated as follows.

Definition 2.6. (Liu [3]) Let \( T \) be a totally ordered set (e.g. time) and let \( (\Gamma, \mathcal{L}, M) \) be an uncertainty space. An uncertain process is a function \( X_t(\gamma) \) from \( T \times (\Gamma, \mathcal{L}, M) \) to the set of real numbers such that \( \{X_t \in B\} \) is an event for any Borel set \( B \) of real numbers at each time \( t \).

Definition 2.7. (Liu [24]) Uncertain processes \( X_{1t}, X_{2t}, \ldots, X_{nt} \) are said to be independent if for any positive integer \( k \) and any times \( t_1, t_2, \ldots, t_k \), the uncertain vectors

\[
\xi_i = (X_{i1t}, X_{i2t}, \ldots, X_{int}), \quad i = 1, 2, \ldots, n
\]

are independent, i.e., for any Borel sets \( B_1, B_2, \ldots, B_n \) of \( k \)-dimensional real vectors, we have

\[
M\left(\bigcap_{i=1}^{n} \{\xi_i \in B_i\}\right) = \prod_{i=1}^{n} M\{\xi_i \in B_i\}.
\]

An uncertain process \( X_t \) is said to have independent increments if

\[
X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}
\]

are independent uncertain variables where \( t_0 \) is the initial time and \( t_1, t_2, \ldots, t_k \) are any times with \( t_0 < t_1 < \cdots < t_k \). An uncertain process \( X_t \) is said to have stationary increments if, for any given \( t > 0 \), the increments \( X_{t+s} - X_s \) are identically distributed uncertain variables for all \( s > 0 \).

Definition 2.8. (Liu [2]) An uncertain process \( C_t \) is said to be a Liu process if

(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,

(ii) \( C_t \) has stationary and independent increments,

(iii) every increment \( C_{t+s} - C_s \) is a normal uncertain variable with an uncertainty distribution

\[
\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}, \quad x \in \mathbb{R},
\]

Definition 2.9. (Liu [2]) Let \( X_t \) be an uncertain process and let \( C_t \) be a Liu process. For any partition
of closed interval \([a, b]\) with \(a = t_1 < t_2 < \cdots < t_{k+1} = b\), the mesh is written as
\[
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
\]

Then Liu integral of \(X_t\) with respect to \(C_t\) is
\[
\int_a^b X_t \, dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),
\]
provided that the limit exists almost surely and is finite. In this case, the uncertain process \(X_t\) is said to be integrable.

**Definition 2.10.** (Liu [22]) Let \(C_t\) be a Liu process and let \(Z_t\) be an uncertain process. If there exist uncertain processes \(\mu_t\) and \(\sigma_t\) such that
\[
Z_t = Z_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dC_s
\]
for any \(t \geq 0\), then \(Z_t\) is said to be differentiable and has an uncertain differential
\[
dZ_t = \mu_t \, dt + \sigma_t \, dC_t.
\]

**Theorem 2.2.** (Yao et al. [28]) Let \(C_t\) be a Liu process. Then there exists an uncertain variable \(K\) such that \(K(\gamma)\) is a Lipschitz constant of the sample path \(C_t(\gamma)\) for each \(\gamma\)
\[
\lim_{x \to +\infty} \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} = 1
\]
and
\[
\mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} \geq 2 \left(1 + \exp \left(-\frac{\pi y}{\sqrt{3}}\right)\right)^{-1} - 1.
\]

**Definition 3.1.** Let \(X_t\) and \(Y_t\) be two solutions of the uncertain differential equation with jumps
\[
dX_t = f(t, X_t) \, dt + g(t, X_t) \, dC_t + h(t, X_t) \, dN_t
\]
with different initial values \(X_0\) and \(Y_0\), respectively. Then the uncertain differential equation with jumps (1) is said to be stable in \(p\)-th moment if
\[
\lim_{|X_0 - Y_0| \to 0} \mathbb{E} \left[ \sup_{t \geq 0} |X_t - Y_t|^p \right] = 0, \quad p > 0.
\]

**Example 3.1.** Consider the following uncertain differential equation with jumps
\[
dX_t = t^2 \, dt + \sigma \, dC_t + \nu \, dN_t
\]
where \(\sigma, \nu\) are positive numbers. The two solutions with different initial values \(X_0\) and \(Y_0\) are
\[
X_t(\gamma) = X_0 + \frac{t^3}{3} + \sigma C_t(\gamma) + \nu N_t(\gamma)
\]
and
\[
Y_t(\gamma) = Y_0 + \frac{t^3}{3} + \sigma C_t(\gamma) + \nu N_t(\gamma)
\]
respectively. Since
\[
|X_t(\gamma) - Y_t(\gamma)| = |X_0 - Y_0|, \quad \forall t \geq 0, \quad \gamma \in \Gamma,
\]
we have
\[ \sup_{t \geq 0} |X_t(\gamma) - Y_t(\gamma)|^p = |X_0 - Y_0|^p, \quad \gamma \in \Gamma. \]

So we can get
\[ \lim_{|X_0 - Y_0| \to 0} \mathbb{E} \left[ \sup_{t \geq 0} |X_t - Y_t|^p \right] = 0, \quad \forall p > 0. \]

Thus, the uncertain differential equation with jumps (2) is stable in \( p \)-th moment.

**Example 3.2.** Consider the following uncertain differential equation with jumps
\[ dX_t = tX_t dt + \sigma X_t dC_t + \nu X_t dN_t, \quad (3) \]
where \( \sigma, \nu \) are positive real numbers. The two solutions with different initial values \( X_0 \) and \( Y_0 \) are
\[ X_t(\gamma) = X_0 \cdot \exp \left( \frac{t^2}{2} + \sigma C_t(\gamma) \right) \cdot (1 + \nu)^N_t(\gamma), \quad \forall \gamma \in \Gamma \]
and
\[ Y_t(\gamma) = Y_0 \cdot \exp \left( \frac{t^2}{2} + \sigma C_t(\gamma) \right) \cdot (1 + \nu)^N_t(\gamma), \quad \forall \gamma \in \Gamma \]
respectively. Since
\[ |X_t(\gamma) - Y_t(\gamma)| = |X_0 - Y_0| \cdot \exp \left( \frac{t^2}{2} + \sigma C_t(\gamma) \right) \cdot (1 + \nu)^N_t(\gamma) \to \infty, \]
as \( t \to \infty \) for each sample \( \gamma \) with \( C_t(\gamma) > 0 \), we can get
\[ \lim_{|X_0 - Y_0| \to 0} \mathbb{E} \left[ \sup_{t \geq 0} |X_t - Y_t|^p \right] = \infty, \quad \forall p > 0. \]

Thus, the uncertain differential equation with jumps (3) is not stable in \( p \)-th moment.

**Theorem 3.1.** The uncertain differential equation with jumps
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t + h(t, X_t)dN_t \]
is stable in \( p \)-th moment if the coefficients \( f(t, x) \), \( g(t, x) \) and \( h(t, x) \) satisfy the strong Lipschitz condition
\[ |f(t, x) - f(t, y)| \leq L_1(t)|x - y|, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0, \]
\[ |g(t, x) - g(t, y)| \leq L_2(t)|x - y|, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0 \]
and
\[ |h(t, x) - h(t, y)| \leq L_3(t)|x - y|, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0, \]
respectively, where \( L_1(t) \) and \( L_2(t) \) are integrable function on \([0, +\infty)\) satisfying
\[ \int_0^{+\infty} L_1(s) ds < +\infty \]
and
\[ \int_0^{+\infty} L_2(s) ds < \frac{\pi}{\sqrt{3}}, \]
and \( L_3(t) \) is monotone and integrable function on \([0, +\infty)\) satisfying
\[ \int_0^{+\infty} L_3(s) ds < +\infty. \]

**Proof.** Assume that \( X_t \) and \( Y_t \) are the solutions of the uncertain differential equation with jumps (1) with different initial values \( X_0 \) and \( Y_0 \), respectively. Then for any Lipschitz continuous sample path \( C_t(\gamma) \), we have
\[ X_t(\gamma) = X_0 + \int_0^t f(s, X_s(\gamma)) ds \]
\[ + \int_0^t g(s, X_s(\gamma)) dC_s(\gamma) \]
\[ + \int_0^t h(s, X_s(\gamma)) dN_s(\gamma), \quad \forall \gamma \in \Gamma, \]
\[ Y_t(\gamma) = Y_0 + \int_0^t f(s, Y_s(\gamma)) ds \]
\[ + \int_0^t g(s, Y_s(\gamma)) dC_s(\gamma) \]
\[ + \int_0^t h(s, Y_s(\gamma)) dN_s(\gamma), \quad \forall \gamma \in \Gamma, \]
By the strong Lipschitz condition, we have
\[ |X_t(\gamma) - Y_t(\gamma)| \]
\[ \leq |X_0 - Y_0| + \int_0^t |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| ds \]
\[ + \int_0^t |g(s, X_s(\gamma)) - g(s, Y_s(\gamma))| \cdot |dC_s(\gamma)| \]
\[ + \int_0^t |h(s, X_s(\gamma)) - h(s, Y_s(\gamma))| dN_s(\gamma) \]
\[ \leq |X_0 - Y_0| + \int_0^t L_1(s)|X_s(\gamma) - Y_s(\gamma)| ds \]
\[ +K(\gamma) \int_0^t L_2(s)X_s(\gamma) - Y_s(\gamma) \, ds \]
\[ + \int_0^t L_3(s)X_s(\gamma) - Y_s(\gamma) \, dN_s(\gamma) \]
\[ = |X_0 - Y_0| + \int_0^t (L_1(s) + K(\gamma)L_2(s)) X_s(\gamma) - Y_s(\gamma) \, ds \]
\[ + \int_0^t L_3(s)X_s(\gamma) - Y_s(\gamma) \, dN_s(\gamma), \]

where \( K(\gamma) \) is the Lipschitz constant of \( C_i(\gamma) \). Let \( \xi_1, \xi_2, \cdots \) denote the iid positive uncertain interarrival times of the uncertain renewal process \( N_t \). Write \( S_0 = 0 \) and \( S_i = \xi_1 + \xi_2 + \cdots + \xi_i \) for \( i \geq 1 \). Then By the Grönwall’s inequality, we get
\[ |X_t(\gamma) - Y_t(\gamma)| \]
\[ \leq |X_0 - Y_0| \cdot \exp \left( \int_0^t (L_1(s) + K(\gamma)L_2(s)) \, ds \right) \]
\[ \cdot \prod_{i=1}^{N_t(\gamma)} (1 + L_3(S_i(\gamma))) \]
\[ \leq |X_0 - Y_0| \cdot \exp \left( \int_0^{+\infty} (L_1(s) + K(\gamma)L_2(s)) \, ds \right) \]
\[ \cdot \prod_{i=1}^{+\infty} (1 + L_3(S_i(\gamma))) \]
\[ \leq |X_0 - Y_0| \cdot \exp \left( \int_0^{+\infty} (L_1(s) + K(\gamma)L_2(s)) \, ds \right) \]
\[ \cdot \exp \left( \sum_{i=1}^{+\infty} L_3(S_i(\gamma)) \right) \]

for all \( t \geq 0 \), and we have
\[ |X_t(\gamma) - Y_t(\gamma)|^p \leq |X_0 - Y_0|^p \]
\[ \cdot \exp \left( p \int_0^{+\infty} (L_1(s) + K(\gamma)L_2(s)) \, ds \right) \]
\[ \cdot \exp \left( p \sum_{i=1}^{+\infty} L_3(S_i(\gamma)) \right). \]

Therefore
\[ \sup_{i \geq 0} |X_t - Y_t|^p \leq |X_0 - Y_0|^p \]
\[ \cdot \exp \left( p \int_0^{+\infty} (L_1(s) + KL_2(s)) \, ds \right) \]
\[ \cdot \exp \left( p \sum_{i=1}^{+\infty} L_3(S_i) \right) \]

almost surely, where \( K \) is a nonnegative uncertain variable such that
\[ \lim_{x \to +\infty} M \{ \gamma \in \Gamma | K(\gamma) \leq x \} = 1 \]

by Theorem 2.2. For any given number \( \varepsilon > 0 \), there exists a real number \( H \) such that the uncertain event
\[ \Lambda = \bigcap_{i=1}^{+\infty} \{ \gamma | \xi_i(\gamma) \geq H \} \]

has an uncertain measure \( M(\Lambda) = 1 - \varepsilon \). Since \( L_3(t) \) is a monotone and integrable function on \([0, +\infty)\), we have
\[ \sum_{i=1}^{+\infty} L_3(S_i(\gamma)) \leq \frac{1}{H} \int_0^{+\infty} L_3(s) \, ds < +\infty, \]

for any \( \gamma \in \Lambda \). So we have
\[ \exp \left( p \sum_{i=1}^{+\infty} L_3(S_i) \right) < +\infty. \]

Since \( \int_0^{+\infty} L_1(s) \, ds < +\infty \), we have
\[ \exp \left( p \int_0^{+\infty} L_1(s) \, ds \right) < +\infty. \]

Taking expected value on both sides for Equation (4), we have
\[ E \left[ \sup_{i \geq 0} |X_t - Y_t|^p \right] \]
\[ \leq |X_0 - Y_0|^p \cdot \exp \left( p \int_0^{+\infty} L_1(s) \, ds \right) \]
\[ \cdot \exp \left( p \sum_{i=1}^{+\infty} L_3(S_i) \right) \]
\[ \cdot \exp \left( p \int_0^{+\infty} KL_2(s) \, ds \right). \]
By Theorem 2.2, we have

\[
E \left[ \exp \left( pK \int_0^{+\infty} L_2(s) ds \right) \right] \\
= \int_0^{+\infty} \mathcal{M} \left\{ \exp \left( pK \int_0^{+\infty} L_2(s) ds \right) \geq x \right\} dx \\
= 1 + \int_0^{+\infty} L_2(s) ds \int_0^{+\infty} \exp \left( py \int_0^{+\infty} L_2(s) ds \right) \mathcal{M} \{ K \geq y \} dy \\
\leq 1 + \int_0^{+\infty} L_2(s) ds \int_0^{+\infty} \exp \left( py \int_0^{+\infty} L_2(s) ds \right) \left\{ 1 - \left( 2 + \exp \left( -\frac{\pi y}{\sqrt{3}} \right) \right)^{-1} - 1 \right\} dy \\
= 1 + 2p \int_1^{+\infty} \left( 1 + x e^{3 \int_0^{+\infty} L_2(s) ds} \right)^{-1} dx \\
< +\infty.
\]

Thus, we can get

\[
\lim_{|X_0 - Y_0| \to 0} E \left[ \sup_{t \geq 0} |X_t - Y_t|^p \right] \\
\leq \lim_{|X_0 - Y_0| \to 0} |X_0 - Y_0|^p \cdot \exp \left( p \int_0^{+\infty} L_1(s) ds \right) \cdot \exp \left( p \sum_{i=1}^{+\infty} L_3 \left( S_i \right) \right) \cdot E \left[ \exp \left( p \int_0^{+\infty} KL_2(s) ds \right) \right] \\
= 0.
\]

Therefore, following the definition of stability in \(p\)-th moment, the uncertain differential equation with jumps is stable in \(p\)-th moment under the strong Lipschitz condition. The theorem is proved. \(\square\)

**Example 3.3.** Consider the following uncertain differential equation with jumps

\[
dX_t = \nu dt + \sigma dC_t + v dN_t. \tag{5}
\]

Note that \(f(t, x) = t, g(t, x) = \sigma\) and \(h(t, x) = \nu\) satisfy the strong Lipschitz condition in Theorem 3.1.

From Theorem 3.1, the uncertain differential equation with jumps (6) is stable in \(p\)-th moment.

**Example 3.4.** Consider the following uncertain differential equation with jumps

\[
dX_t = \nu dt + \exp(-t - X_t) dC_t + \exp(-t^2) X_t dN_t. \tag{6}
\]

Note that \(f(t, x) = t, g(t, x) = \exp(-t - x)\) and \(h(t, x) = \exp(-t^2) x\) satisfy

\[
|f(t, x) - f(t, y)| = 0, \forall x, y \in \mathbb{R}, t \geq 0, \\
|g(t, x) - g(t, y)| \leq \exp(-t)|x - y|, \forall x, y \in \mathbb{R}, t \geq 0
\]

and

\[
|h(t, x) - h(t, y)| \leq \exp(-t^2)|x - y|, \\
\forall x, y \in \mathbb{R}, t \geq 0
\]

respectively. Since \(\exp(-t^2)\) is a monotone function of \(t\) and

\[
\int_0^{+\infty} \exp(-t) dt = 1 < \frac{\pi}{\sqrt{3}}, \\
\int_0^{+\infty} \exp(-t^2) dt = \frac{\sqrt{\pi}}{2} < +\infty,
\]

by Theorem 3.1, the uncertain differential equation with jumps (6) is stable in \(p\)-th moment.

**Remark.** In fact, Theorem 3.1 gives a sufficient condition for uncertain differential equation with jumps being stable in \(p\)-th moment.

**Theorem 3.2.** The linear uncertain differential equation with jumps

\[
dX_t = (\mu_1 X_t + \mu_2 t) dt + (\sigma_1 X_t + \sigma_2 t) dC_t + (\nu_1 X_t + \nu_2 t) dN_t \tag{7}
\]
is stable in $p$-th moment if the real functions
\[ \mu_{i1}, \sigma_{i1}, v_{i1}, i = 1, 2 \]
satisfy
\[
\int_0^{+\infty} |\mu_{11}| dt < +\infty, \int_0^{+\infty} |\sigma_{11}| dt < +\infty,
\]
and \[ |v_{11}| \]
is a monotone function of $t$.

**Proof.** Note that 
\[ f(t, x) = \mu_{11} x + \mu_{21}, \quad g(t, x) = \sigma_{11} x + \sigma_{21}, \quad h(t, x) = v_{11} x + v_{21}. \]
Then we have
\[
|f(t, x) - f(t, y)| = |\mu_{11}| \cdot |x - y|, \\
g(t, x) - g(t, y)| = |\sigma_{11}| \cdot |x - y|, \\
h(t, x) - h(t, y)| = |v_{11}| \cdot |x - y|.
\]
Taking \[ L_1(t) = |\mu_{11}|, \quad L_2(t) = |\sigma_{11}|, \quad L_3(t) = |v_{11}|, \]
we have
\[
\int_0^{+\infty} L_1(t) dt < +\infty, \quad \int_0^{+\infty} L_2(t) dt < \frac{\pi}{\sqrt{3}}, \\
\int_0^{+\infty} L_3(t) dt < +\infty,
\]
and \[ L_3(t) \]
is a monotone function of $t$. By Theorem 3.1, the linear uncertain differential Equation (7) is stable in $p$-th moment. The theorem is proved. \(\Box\)

**Example 3.5.** Consider the following linear uncertain differential equation with jumps
\[
dX_t = \exp(-t)X_t dt + \exp(-t^2)X_t dC_t \\
+ \left( \exp(-t^2)X_t + t \right) dN_t. \quad (8)
\]

Note that \[ \mu_{11} = \exp(-t), \quad \sigma_{11} = \exp(-t^2) \] and \[ v_{11} = \exp(-t^2) \]
satisfy
\[
\int_0^{+\infty} |\mu_{11}| dt = \int_0^{+\infty} \exp(-t) dt = 1 < \frac{\pi}{\sqrt{3}}, \\
\int_0^{+\infty} |\sigma_{11}| dt = \int_0^{+\infty} \exp(-t^2) dt = \frac{1}{2} < \frac{\pi}{\sqrt{3}}, \\
\int_0^{+\infty} |v_{11}| dt = \int_0^{+\infty} \exp(-t^2) dt = \frac{\sqrt{\pi}}{2} < +\infty,
\]
and \[ |v_{11}| = \exp(-t^2) \]
is a monotone function of $t$. By Theorem 3.2, the uncertain differential equation with jumps (8) is stable in $p$-th moment.

4. **Comparison of stability**

In this section, the relationship between the stability in measure and the stability in $p$-th moment for an uncertain differential equation with jumps is discussed.

**Theorem 4.1.** If the uncertain differential equation with jumps
\[
dX_t = f(t, X_t) dt + g(t, X_t) dC_t + h(t, X_t) dN_t
\]
is stable in $p$-th moment, then it is stable in measure.

**Proof.** From Definition 3.1, for two solutions $X_t$ and $Y_t$ with different initial values $X_0$ and $Y_0$, respectively, we have
\[
\lim_{|X_0-Y_0| \to 0} E \left[ \sup_{t\geq 0} |X_t - Y_t|^p \right] = 0, \quad p > 0.
\]

By Markov inequality, for any given real number $\varepsilon > 0$, we have
\[
\lim_{|X_0-Y_0| \to 0} M \left\{ \sup_{t\geq 0} |X_t - Y_t| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon^p} \cdot \lim_{|X_0-Y_0| \to 0} E \left[ \sup_{t\geq 0} |X_t - Y_t|^p \right] = 0.
\]

So we can get
\[
\lim_{|X_0-Y_0| \to 0} M \left\{ \sup_{t\geq 0} |X_t - Y_t| \leq \varepsilon \right\} = 1,
\]
for any given real number $\varepsilon > 0$.

Therefore, from Definition 2.12, the uncertain differential equation with jumps (1) is stable in measure. The theorem is proved. \(\Box\)

**Example 4.1.** Consider the following uncertain differential equation with jumps
\[
dX_t = \exp(-t^2) dt + \exp(-t - X_t) dC_t \\
+ \exp(-t^2 - X_t) dN_t. \quad (9)
\]

Note that \[ f(t, x) = \exp(-t^2), \quad g(t, x) = \exp(-t - x) \] and \[ h(t, x) = \exp(-t^2 - x) \].

On the one hand, we have
\[
|f(t, x) - f(t, y)| = 0, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0, \\
|g(t, x) - g(t, y)| \leq \exp(-t)|x - y|, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0.
\]
and
\[ |h(t, x) - h(t, y)| \leq \exp(-t^2)|x - y|, \]
\[ \forall x, y \in \mathbb{R}, t \geq 0, \]
respectively. Since \( \exp(-t^2) \) is a monotone function of \( t \) and
\[ \int_0^{+\infty} \exp(-t)dt = 1 < \frac{\pi}{\sqrt{3}}, \]
\[ \int_0^{+\infty} \exp(-t^2)dt = \frac{\sqrt{\pi}}{2} < +\infty, \]
by Theorem 3.1, the uncertain differential equation with jumps (9) is stable in \( p \)-th moment.

On the other hand, the coefficient \( f(t, x) \) and \( g(t, x) \) satisfy
\[ |f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq \exp(-t)|x - y|, \]
\[ \forall x, y \in \mathbb{R}, t \geq 0, \]
where \( \exp(-t) \) is integrable on \( [0, +\infty) \), and the coefficient \( h(t, x) \) satisfies
\[ |h(t, x) - h(t, y)| \leq \exp(-t^2)|x - y|, \]
\[ \forall x, y \in \mathbb{R}, t \geq 0, \]
where \( \exp(-t^2) \) is monotone and integrable on \( [0, +\infty) \). By Theorem 2.3, the uncertain differential equation with jumps (9) is stable in measure.

5. Conclusion

This paper studied the stability in \( p \)-th moment for the uncertain differential equation with jumps, and gave a sufficient condition for the uncertain differential equation with jumps being stable in \( p \)-th moment. Then, the relationship between the stability in measure and the stability in \( p \)-th moment was discussed.

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References


