Valuation of stock loan under uncertain mean-reverting stock model

Gang Shi\textsuperscript{a}, Zhiqiang Zhang\textsuperscript{b} and Yuhong Sheng\textsuperscript{c,}\textsuperscript{*}
\textsuperscript{a}Department of Computer Sciences, Tsinghua University, Beijing, China
\textsuperscript{b}School of Mathematics and Computer Science, Shanxi Datong University, Datong, China
\textsuperscript{c}College of Mathematical and System Sciences, Xinjiang University, Urumchi, China

Abstract. Stock loan is different from the traditional loan, it needs to be collateralized by stock. Fairly valuing stock loan is very important for financial market participants. The main contribution of this paper is to give a valuing method of stock loan in uncertain environment. Under the assumption that the underlying stock price follows an uncertain mean-reverting stock model, the price formulas of standard stock loan and capped stock loan are derived by using the method of uncertain calculus. Some numerical examples are presented to illustrate the related results.

Keywords: Stock loan, uncertainty theory, uncertain differential equation, uncertain stock model

1. Introduction

Stock loan is different from the traditional loan in which the stocks are employed as the only guarantee. This type of loan is a contract between a borrower and a bank. If a borrower obtains some money from a bank with his or her stocks as collateral, this contract gives the borrower the right rather than the obligation to regain his or her stocks at any time before the loan maturity by repaying the bank the principal plus the loan interest. This type of financial products can be used as a hedging tool against the letting down of stock market for the borrowers. When the price of the stock goes down, the borrower can choose to give up the collateral rather than to regain the stock to avoid the loss from devaluation of the stock. On the other hand, if the stock price goes up, he or she can choose to redeem the stock by repaying the bank the loan amount and the loan interest. Another advantage is reflected in such an situation that the borrower needs money urgently but he or she is unwilling to lose his or her ownership of stocks completely.

The associated research of stock loan was pioneered by Xia and Zhou [30]. They derived a closed-form pricing formula of stock loan based on the classical Black-Scholes model [2] by using a purely probabilistic approach. Then Zhang and Zhou [35] extended their framework to a problem of valuation of stock loans with regime switching model and gave the stock loan pricing formulas for this type of model. Afterwards, the problems of stock loan pricing were investigated by many scholars, including Liang, Wu and Jiang [11], Wong and Wong [29], Pascucci, Suarez-Taboada and Vazquez [24] and Cai and Sun [3], and so on.

The previous researches on stock loans valuation are all within the framework of probability theory. The stock loans pricing problem were solved by using probabilistic pricing approach based on the assumption that the stock price follows the stochastic differential equations. However, this assumption was challenged by many scholars. Liu [17] proposed a paradox...
that gave a convincing explanation to show that using stochastic differential equations to describe stock price is inappropriate. Kahneman and Tversky [9] showed that the probability itself is not served as the basis of decision making by investors, and investors usually make a nonlinear transformation of probability as their basis which they based on to make decisions. Liu [18] expressed the view that human beings usually estimate a much wider range of values than the object actually takes. In real financial practice, investors’ belief degrees usually play an important role in decision making and influence the financial market performance. The belief degrees behave neither like randomness nor like fuzziness, and it can not be described with probability theory and fuzzy theory (see Liu [16]). For rationally dealing with human’s belief degrees, Liu [12] founded uncertainty theory in 2007 and refined it in 2010. For modeling the evolution of phenomena with uncertainty, Liu [13] proposed the concept of uncertain process, and established uncertain calculus to deal with differentiation and integration of uncertain processes.

Uncertainty theory was first introduced into the study of finance by Liu [14] in 2009. Different from Black-Scholes framework, Liu [14] proposed an uncertain stock model and gave the European option price formula in which the stock price is described by an uncertain differential equation. Chen [4] derived the American option price formula, Zhang and Liu [36] verified the geometric average Asian option price formulas, and Zhang, Liu and Sheng [38] gave the formulas of power option for Liu’s uncertain stock model, respectively. Peng and Yao [25] extended Liu’s stock model to the case of stock model with mean-reverting process, and Yao [33] proved a no-arbitrage theorem for uncertain stock model. Chen, Liu and Ralescu [6] considered the case of stock with dividends and proposed an uncertain stock model with periodic dividends and derived the pricing formulas of some options under this model. Besides, Chen and Gao [5] investigated the interest term structure within the framework of uncertainty theory and obtained the zero-coupon bond price formula for uncertain interest rate model. Zhang, Ralescu and Liu [37] discussed the pricing problem of interest rate ceiling and floor for uncertain financial market. Research on currency option within the framework of uncertainty theory began with Liu, Chen and Ralescu [20] in which uncertain differential equations were employed to model currency exchange rate and some currency option price formulas were derived. Zhang, Liu and Ding [39] firstly discussed the problem of stock loan valuation within the framework of uncertainty theory, and gave the pricing formulas of stock loan for Liu’s uncertain stock model. Considering the stock prices fluctuate around some average level in long run, we will investigate the valuation of stock loan under uncertain mean-reverting stock model. The rest of the paper is organized as follows. In next section, some useful concepts and theorems of uncertainty theory as needed are introduced. In Section 3, the valuation of the standard stock loan for uncertain mean-reverting stock model is investigated. In Section 4, we explore the pricing problem of capped stock loan for this type of stock model. In Section 5, valuing stock loan for general uncertain stock loan is discussed. Finally, a brief conclusion is given in Section 6.

2. Preliminaries

Definition 2.1. [14] An uncertain process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{t+\delta} - C_t$ is a normal uncertain variable with expected value 0 and variance $t^2$.

Definition 2.2. [13] Suppose $C_t$ is a Liu process, and $f$ and $g$ are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

(2.1)

is called an uncertain differential equation.

Definition 2.3. [32] Let $\alpha$ be a number with $0 < \alpha < 1$. An uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

(2.2)

is said to have an $\alpha$-path $X^\alpha_t$ if it solves the corresponding ordinary differential equation

$$dX^\alpha_t = f(t, X^\alpha_t)dt + |g(t, X^\alpha_t)|\Phi^{-1}(\alpha)dt$$

(2.3)

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \quad (2.4)$$
Theorem 2.1. [32] Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (2.5) \]
respectively. Then for any time $\alpha$
\[ \mathcal{M}\{ X_t \leq X_t^\alpha, \forall t \} = \alpha, \quad (2.6) \]
\[ \mathcal{M}\{ X_t > X_t^\alpha, \forall t \} = 1 - \alpha. \quad (2.7) \]

Theorem 2.2. [32] Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (2.8) \]
respectively. Then the solution $X_t$ has an inverse uncertainty distribution
\[ \Psi_t^{-1}(\alpha) = X_t^\alpha. \quad (2.9) \]

Theorem 2.3. [31] Let $X_t$ and $X_t^\alpha$ be the solution and $\alpha$-path of the uncertain differential equation
\[ dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (2.10) \]
respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the supremum
\[ \sup_{0 \leq t \leq s} J(X_t) \]
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} J(X_t^\alpha); \quad (2.12) \]
and the infimum
\[ \inf_{0 \leq t \leq s} J(X_t) \]
has an inverse uncertainty distribution
\[ \Psi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} J(X_t^\alpha). \quad (2.14) \]

3. Valuation of stock loan

Liu [14] suggested to describe the stock price process by using an uncertain differential equation and proposed an uncertain stock model as follows
\[ \begin{align*}
    dX_t &= rX_t dt \\
    dS_t &= \mu S_t dt + \sigma S_t dC_t
\end{align*} \quad (3.1) \]
where $X_t$ is the bond price, $S_t$ is the stock price, $r$ is the riskless interest rate, $\mu$ is the log-drift, $\sigma$ is the log-diffusion, and $C_t$ is a Liu process.

Considering the case of the stock price usually fluctuates around some average price in long run, Peng and Yao [25] extended Liu’s uncertain stock model to an uncertain mean-reverting stock model as follows
\[ \begin{align*}
    dX_t &= rX_t dt \\
    dS_t &= (m - \alpha S_t)dt + \sigma dC_t
\end{align*} \quad (3.2) \]
where $X_t$ is the bond price, $S_t$ is the stock price, $r$ is the riskless interest rate, $m$, $\alpha$ and $\sigma$ are constants, and $C_t$ is a Liu process. This type of model can be used to capture price movements that have the tendency to move towards an equilibrium level.

Suppose a borrower can obtain amount $K$ from a bank with one share of his or her stock as collateral. After paying a service fee $c$ ($0 < c < K$) to the bank, the borrower receives the amount $(K - c)$. The borrower has the right to redeem the stock at any time prior to the loan maturity time $T$ by repaying the bank the principal plus interest associated to the loan that is $K \exp(\theta T)$, where $\theta > r$ is the loan interest rate. A basic problem on stock loan is what are the fair value of the principal $K$, the loan interest $\theta$ and the fee $c$ charged by the bank for providing the service. The key for solving this problem is to fairly evaluate the value of the stock loan.

The main objective of this paper is to evaluate the stock loan value, in turn it can be used to determine the rational values of the parameters $K$, $\theta$ and $c$.

From the above description on stock loan, we can see that the stock loan means that the borrower pays $S_0 - (K - c)$ to buy an American option with a time-dependent strike price $K \exp(\theta t)$ and maturity $T$ at time 0. The present value of the payoff of the borrower is
\[ \sup_{0 \leq t \leq T} \exp(-rt) [S_t - K \exp(\theta t)]^+. \quad (3.3) \]

Thus the value of the stock loan should be the expected present value of the payoff. So if we assume a stock loan has loan amount $K$, loan interest rate $\theta$ and loan maturity time $T$, then the value of the stock loan should be
\[ f = E \left[ \sup_{0 \leq t \leq T} \exp(-rt) [S_t - K \exp(\theta t)]^+ \right]. \quad (3.4) \]

Theorem 3.1. Assume a stock loan for the stock model (3.2) has loan amount $K$, loan interest rate $\theta$ and loan maturity time $T$. Then the value of the stock loan is
where \( S_t^0 = \frac{1}{a} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) (1 - \exp(-at)) + \exp(-at)S_0. \)

**Proof.** Solving the ordinary differential equation

\[
dS_t^0 = (m - aS_t^0)dt + \sigma \Phi^{-1}(\alpha)dt
\]

where \( 0 < \alpha < 1 \) and \( \Phi^{-1}(\alpha) \) is the inverse standard normal uncertainty distribution, we have

\[
S_t^0 = \frac{1}{a} \left( m + \sigma \Phi^{-1}(\alpha) \right) (1 - \exp(-at)) + \exp(-at)S_0
\]

\[= \frac{1}{a} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) (1 - \exp(-at)) + \exp(-at)S_0. \]

That means that the uncertain differential equation

\[
dS_t = (m - aS_t)dt + \sigma dC_t
\]

has an \( \alpha \)-path

\[
S_t^0 = \frac{1}{a} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) (1 - \exp(-at)) + \exp(-at)S_0. \]

Since \( J(x) = \exp(-rt)(x - K \exp(\theta t))^+ \) is an increasing function, it follows from Theorem 2.3 that

\[
\sup_{0 \leq t \leq T} J(S_t) = \sup_{0 \leq t \leq T} \exp(-rt)(S_t - K \exp(\theta t))^+ \]

has an inverse uncertainty distribution

\[
\sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^0 - K \exp(\theta t) \right]^+. \]

Therefore, the value of the stock loan is

\[
f = \int_0^1 \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^0 - K \exp(\theta t) \right]^+ \, d\alpha
\]

where \( S_t^0 = \frac{1}{a} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha} \right) (1 - \exp(-at)) + \exp(-at)S_0. \)

**Example 3.1.** Assume the riskless interest rate \( r = 0.06 \), the initial stock price \( S_0 = 40 \), the loan amount \( K = 28 \), loan interest rate \( \theta = 0.065 \), the maturity time \( T = 1 \) and the parameters of the model (3.2) \( m = 32, \alpha = 0.8 \) and \( \sigma = 5.4 \). By the formula of Theorem 3.1, we can calculate out that the value of stock loan is

\[
f = 12.12.
\]

Since the borrower pays \( S_0 - (K - c) \) at time 0, the value of the stock loan satisfies the equation

\[
f = S_0 - (K - c).
\]

Then the fair service charge would be

\[c = 0.12. \]

**4. Valuation of capped stock loan**

In this section, we study the valuation of capped stock loan, in which there is a capped limit for the stock price. For this type of stock loan, the holder will get the stock if the stock price is lower than the capped limit level after he or she refunds the loan, otherwise the money he or she will get is equal to the capped limit, that is the borrower will get the minimum value between the predetermined money and the stock price after paying back to bank the loan. In this case, the possible maximum loss the bank will face is the difference between the predetermined capped level and the accumulative loan amount. So setting up such a capped limit for stock price, the bank can avoid the possible large loss in the future time. Thus capped stock loans has more advantages than standard loans in which the borrower still has the possibility of obtaining a profit, and the bank may cut down future risk in the meantime. There are two types of cap: one is constant cap, another is the cap with a constant growth rate.

Suppose a capped stock loan has loan amount \( K \), loan interest rate \( \theta \), and loan maturity time \( T \). Assume the loan has constant cap \( L \). Then the present value of the payoff of the borrower is

\[
\sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L - K \exp(\theta t)]^+. \]

Thus the value of the capped stock loan should be the expected present value of the payoff. Let \( f \) represent the value of the capped stock loan. Then the value of the capped stock loan is

\[
f = E \left[ \sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L - K \exp(\theta t)]^+ \right].
\]

**Theorem 4.1.** Assume a stock loan for the stock model (3.2) has loan amount \( K \), loan interest rate \( \theta \), constant cap \( L \) and loan maturity time \( T \). Then the value of the capped stock loan is
\[ f = \int_0^1 \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L - K \exp(\theta t) \right]^+ \, d\alpha \tag{4.3} \]

where \( S_t^\alpha = \frac{1}{\alpha} \left( \frac{m}{\alpha} + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left( 1 - \exp(-at) \right) + \exp(-at)S_0 \).

**Proof.** Since \( J(x) = \exp(-rt)[x \wedge L - K \exp(\theta t)]^+ \) is an increasing function, it follows from Theorem 2.3 that \( \sup_{0 \leq t \leq T} J(S_t) = \sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L - K \exp(\theta t)]^+ \) has an inverse uncertainty distribution \( \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L - K \exp(\theta t) \right]^+ \tag{4.4} \).

Therefore the value of the stock loan is

\[ f = \int_0^1 \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L - K \exp(\theta t) \right]^+ \, d\alpha \tag{4.5} \]

where \( S_t^\alpha = \frac{1}{\alpha} \left( \frac{m}{\alpha} + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left( 1 - \exp(-at) \right) + \exp(-at)S_0 \).

**Example 4.1.** Assume the riskless interest rate \( r = 0.06 \), the initial stock price \( S_0 = 40 \), the loan amount \( K = 28 \), loan interest rate \( \theta = 0.065 \), the constant cap \( L = 65 \), the maturity time \( T = 1 \) and the parameters of the model (3.2) \( m = 56, a = 0.8 \) and \( \sigma = 7.5 \). By the formula of Theorem 4.1, we can calculate out that the value of stock loan is

\[ f = 22.57. \]

By the equation \( f = S_0 - (K - c) \), we have the fair service fee

\[ c = 10.57. \]

The cap with a constant growth rate is a time-varying cap that grows at a constant rate \( \beta > 0 \) which actually is a function of time, that is

\[ L_t = L \exp(\beta t). \tag{4.6} \]

Suppose a stock loan with cap \( L_t \) given by (4.6) has loan amount \( K \), loan interest rate \( \theta \), and loan maturity time \( T \). Then the present value of the payoff of the borrower is

\[ \sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L \exp(\beta t) - K \exp(\theta t)]^+. \tag{4.7} \]

Thus the value of the stock loan should be the expected present value of the payoff. Let \( f \) represent the value of this type of capped stock loan. Then the value of the capped stock loan is

\[ f = E \left[ \sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L \exp(\beta t) - K \exp(\theta t)]^+ \right]. \tag{4.8} \]

**Theorem 4.2.** Assume a stock loan for the stock model (3.2) has loan amount \( K \), loan interest rate \( \theta \), constant growth rate cap \( L \exp(\beta t) \) and loan maturity time \( T \). Then the value of the capped stock loan is

\[ f = \int_0^1 \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L \exp(\beta t) - K \exp(\theta t) \right]^+ \, d\alpha \tag{4.9} \]

where \( S_t^\alpha = \frac{1}{\alpha} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left( 1 - \exp(-at) \right) + \exp(-at)S_0 \).

**Proof.** Since \( J(x) = \exp(-rt)[x \wedge L \exp(\beta t) - K \exp(\theta t)]^+ \) is an increasing function, it follows from Theorem 2.3 that \( \sup_{0 \leq t \leq T} J(S_t) = \sup_{0 \leq t \leq T} \exp(-rt)[S_t \wedge L \exp(\beta t) - K \exp(\theta t)]^+ \) has an inverse uncertainty distribution

\[ \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L \exp(\beta t) - K \exp(\theta t) \right]^+. \tag{4.10} \]

Therefore the value of the capped stock loan is

\[ f = \int_0^1 \sup_{0 \leq t \leq T} \exp(-rt) \left[ S_t^\alpha \wedge L \exp(\beta t) - K \exp(\theta t) \right]^+ \, d\alpha \tag{4.11} \]

where \( S_t^\alpha = \frac{1}{\alpha} \left( m + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left( 1 - \exp(-at) \right) + \exp(-at)S_0 \).

**Example 4.2.** Assume the riskless interest rate \( r = 0.06 \), the initial stock price \( S_0 = 40 \), the loan amount \( K = 28 \), loan interest rate \( \theta = 0.065 \), the cap \( L = 65 \) with growth rate \( \beta = 0.05 \), the maturity time \( T = 1 \) and the parameters of the model (3.2) \( m = 56, a = 0.8 \) and \( \sigma = 7.5 \). By the Theorem 4.2, we calculate out that the value of stock loan is

\[ f = 24.6. \]

By the equation \( f = S_0 - (K - c) \), we have the fair service fee

\[ c = 12.6. \]
5. Valuing stock loan for general uncertain stock model

For a general uncertain stock model

\[
\begin{align*}
\frac{dX_t}{X_t} &= r_t dt \\
\frac{dS_t}{S_t} &= F(t, S_t)dt + G(t, S_t)dC_t
\end{align*}
\]

(5.1)

where \(X_t\) is the bond price, \(S_t\) is the stock price, \(r\) is the riskless interest rate, \(F\) and \(G\) are two functions, and \(C_t\) is a Liu process. However, for the general uncertain differential equation

\[
\frac{dS_t}{S_t} = F(t, S_t)dt + G(t, S_t)dC_t,
\]

(5.2)

its analytic solution is usually unreachable. In this case, a numerical method is needed. Yao-Chen Formula (Theorem 2.1) provides a numerical method to solve it via the \(\alpha\)-paths, whose procedure is designed as follows.

**Step 1.** Fix \(\alpha\) on (0,1).

**Step 2.** Solve the ordinary differential equation

\[
\alpha dt = F(t, S_t^\alpha)dt + G(t, S_t^\alpha) \Phi^{-1}(\alpha)dt
\]

(5.3)

via a numerical method where

\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.
\]

(5.4)

**Step 3.** Obtain the \(\alpha\)-path.

By this method, the problem of valuing stock loan for general uncertain stock model can be solved. Compared with the methods in stochastic financial theory, this method is more efficient and effective, and it is convenient to use.

Recently, granular computing is becoming popular to deal with the human-data (see Peters and Weber [26], Livi and Sadeghian [21], Skowron, Jankowski and Dutta [27], and Wilke and Portmann [28]). Liu, Gegov and Cocea [19], and Ahmad and Pedrycz [1] studied the rule-based systems by using granular computing. Maciel, Ballini and Gomide [23] made a granular analytics for interval time series forecasting. Kreinovich [10] gave the method for solving equations (and systems of equations) under uncertainty. The more applications of granular computing, readers may refer to Dubois and Prade [8], Loia et al. [22], Yao [34], and Ciucci [7]. Since granular computing is a very useful technique to deal with the systems of equations under uncertainty, it is worth of future research to use granular computing techniques to solve the stock loan pricing problem in uncertain environments.

6. Conclusion

Stock loan is a type of financial products with complex feature, fairly valuing stock loan is very important and difficult. In this paper, a new pricing method was presented. The pricing problem of stock loan was investigated within the framework of uncertainty theory. Under the assumption that the underlying stock price following an uncertain mean-reverting stock model, the price formulas of standard stock loan and capped stock loan were derived by using the theory of uncertain calculus. The valuation of stock loan with automatic termination clause and margin, and under other uncertain stock models will be investigated in our future research.

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grants Nos. 61563050, 61462086) and Doctoral Fund of Xinjiang University (No. BS150206).

References