Some new ranking criteria in data envelopment analysis under uncertain environment

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ABSTRACT

Data Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision-making units (DMUs), which has been applied extensively to education, hospital, finance, etc. However, in real-world situations, the data of production processes cannot be precisely measured in some cases, which leads to the research of DEA in uncertain environments. This paper will give some researches to uncertain DEA based on uncertainty theory. Due to the uncertain inputs and outputs, we will give three uncertain DEA models, as well as three types of fully ranking criteria. For each uncertain DEA model, its crisp equivalent model is presented to simplify the computation of uncertain models. Finally, a numerical example is presented to illustrate the three ranking criteria.

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1. Introduction

Data envelopment analysis (DEA), as an useful management and decision tool, has been widely used since it was first invented by Charnes, Cooper, and Rhodes (1978). The method is followed by a series of theoretical extensions, such as Banker, Charnes, and Cooper (1984), Charnes, Cooper, Golany, Seiford, and Stutz (1985), Petersen (1990), Tone (2001) and Cooper, Seiford, and Tone (2001). More DEA papers can refer to Seiford (1994) in which 500 references are documented.

In many cases, decision makers are interested in a complete ranking over the dichotomized classification. The researches on ranking have come up for this reason. Over the last decade, many literatures on ranking in DEA have been published. By evaluating DMUs through both self and peer pressure, Sexton, Silkman, and Hogan (1986) can attain a more balanced view of the decision-making units. Andersen and Petersen (1993) developed the super-efficiency approach to get a ranking value which may be greater than one through evaluated DMU's exclusion from the linear constraints. In the benchmark ranking method (Torgersen, Forsund, & Kittelsen, 1996), a DMU is highly ranked if it is chosen as a useful target for many other DMUs.

Most methods of ranking DMUs assume that all inputs and outputs data are exactly known. However, in real situations, such as in a manufacturing system, a production process or a service system, inputs and outputs are volatile and complex so that they are difficult to measure in an accurate way. Thus, people tend to use fuzzy theory to describe the indeterminate inputs and outputs, which motivates the fuzzy DEA. Generally speaking, fuzzy DEA method can be categorized into four types: the tolerance approach, the $a$-level based approach, the fuzzy ranking approach and the possibility approach (Adel, Emrouznejad, & Tavana, 2011). In the tolerance approach (refer to Sengupta (1992)), tolerance levels on constraint violations are defined to integrate fuzziness into the DEA models, and the input and output coefficients can be thus treated as criss. The $a$-level based approach may be the most popular model of fuzzy DEA. This method discretize the original problem into a series of parametric programs in order to decide the $a$-cuts of the membership function of efficiency. Related studies include Kao and Liu (2000), Entani, Maeda, and Tanaka (2002), Liu (2008) and Angiz, Emrouznejad, and Mustafa (2012), etc. The fuzzy ranking model is first proposed by Guo and Tanaka (2001), and it focus on determining the fuzzy efficiency scores of DMUs using optimization methods which require ranking fuzzy sets. One can also refer to León, Liern, Ruiz, and Sirvent (2003), Wang and Luo (2006) or Angiz, Tajaddini, Mustafa, and Kamali (2012) for more concepts and information of the fuzzy ranking method. In the possibility approach, the fuzzy DEA models are converted to possibility linear program problem by using possibility
measures. See Lertworasirikul, Fang, Joines, and Nuttle (2003) for example. Other studies on fuzzy DEA include fuzzy goal programming method (Sheth & Konstantinos, 2003), fuzzy random DEA model in hybrid uncertain environments (Qin & Liu, 2010), fuzzy rough DEA model (Shiraz, Charles, & Jalalzadeh, 2014), cross evaluation approach (Costantino, Dotoli, Epicoco, Falagario, & Sciancalepore, 2012), and fuzzy clustering approach (David & Deep, 2012), etc.

Although the fuzzy DEA models are popular and in most time effective, it may bring some problems to the decision makers in some cases. This is because the possibility measure defined in fuzzy theory doesn’t satisfy duality, as explained in Liu (2012). For this reason, an uncertainty theory was founded by Liu (2007) in 2007, and refined by Liu (2010a) in 2010 to deal with the people’s belief degree mathematically. A concept of uncertain variable is used to model uncertain quantity, and belief degree is regarded as its uncertainty distribution. As extensions of uncertainty theory, uncertain programming was proposed by Liu (2007), and refined by Liu (2010b; Zeng, Wen, & Kang, 2013), uncertain graph (Gao, 2013; Gao & Gao, 2013), etc. As an application, this work was followed by uncertain multiobjective programming models, uncertain goal programming models (Liu & Chen, 2013), and uncertain multi-level programming models (Liu & Yao, 2010).

In this paper, we will assume the inputs and outputs in DEA models are uncertain variables, and introduce some new DEA models and their ranking criteria based on uncertainty theory. The remainder of this paper is organized as follows: Some basic concepts and results on uncertainty theory will be introduced in Section 2; Section 3 will give some basic introduction to DEA models; The method to obtain uncertainty distribution is introduced in Section 4. In Section 5, we will give three uncertain DEA models, three fully ranking criteria, as well as their equivalent deterministic models. Finally, a numerical example will be given to illustrate the uncertain DEA model and the ranking method in Section 6.

2. Preliminaries

Uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010a) in 2010. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty. In this section, we will state some basic concepts and results on uncertain variables. These results are crucial for the remainder of this paper.

Let $\Gamma$ be a nonempty set, and $\mathcal{L}$ a $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in \mathcal{L}$ is assigned a number $M(\Lambda) \in [0, 1]$. In order to ensure that the number $M(\Lambda)$ has certain mathematical properties, Liu (2007, 2010a) presented the three axioms:

(i) $M(\Gamma) = 1$ for the universal set $\Gamma$.

(ii) $M(\{\Lambda\}) + M(\{\Lambda'\}) = 1$ for any event $\Lambda$.

(iii) For any countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

\[
M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).
\]

The triplet $(\Gamma, \mathcal{L}, M)$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu (2012), thus producing the fourth axiom of uncertainty theory:

(iv) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \ldots, \infty$. Then the product uncertain measure $M$ is an uncertain measure satisfying

\[
M\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \prod_{k=1}^{\infty} M_k(\Lambda_k).
\]

An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers (Liu, 2007). In order to describe an uncertain variable in practice, the concept of uncertainty distribution is defined as

\[
\Phi(x) = M(\{\xi \leq x\})
\]

for any real number $x$. For example, the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x-a)/(b-a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b.
\end{cases}
\]

An uncertain variable $\xi$ is called zigzag if it has a zigzag uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x-a)/(2b-a), & \text{if } a \leq x \leq b \\
(x+c-2b)/(2c-b), & \text{if } b \leq x \leq c \\
1, & \text{if } x \geq c
\end{cases}
\]

denoted by $\mathcal{Z}(a, b, c)$ where $a$, $b$, $c$ are real numbers with $a < b < c$. An uncertain variable $\xi$ is called normal if it has a normal uncertainty distribution

\[
\Phi(x) = \left(1 + \exp\left(-\frac{\pi(x-a)}{\sqrt{3}\sigma}\right)\right)^{-1}
\]

denoted by $\mathcal{N}(e, \sigma)$ where $e$ and $\sigma$ are real numbers with $\sigma > 0$. An uncertainty distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}(x)$ exists and is unique for each $x \in (0, 1)$. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

\[
M\left(\bigcap_{i=1}^{n} (\xi_i \in B_i)\right) = \prod_{i=1}^{n} M(\xi_i \in B_i)
\]

for any Borel sets $B_1, B_2, \ldots, B_n$.

Theorem 1 (Liu, 2010a). Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then

\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]

is an uncertain variable with inverse uncertainty distribution

\[
\Psi^{-1} = f(\Phi_1^{-1}(x), \Phi_2^{-1}(x), \ldots, \Phi_n^{-1}(x)).
\]

Theorem 2 (Liu & Ha, 2010). Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\zeta = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

\[
E[\zeta] = \int_{0}^{1} f(\Phi_1^{-1}(x), \ldots, \Phi_n^{-1}(x), \Phi_{m+1}^{-1}(1-x), \ldots, \Phi_n^{-1}(1-x))dx
\]

provided that $E[\zeta]$ exists.
3. DEA model

CCR model is one of the most frequently used DEA model, which was proposed by Charnes et al. (1978). Since the following sections will use this model, we will give some basic introduction to CCR model. Firstly let us review some symbols and variables:

\( \text{DMU}_k; \) the \( k \)th DMU, \( k = 1, 2, \ldots, n; \)
\( \text{DMU}_k; \) the target DMU;
\( x_{ij} \in \mathbb{R}^n; \) the inputs vector of DMU, \( k = 1, 2, \ldots, n; \)
\( x_0 \in \mathbb{R}^n; \) the inputs vector of the target DMU;
\( y_{ij} \in \mathbb{R}^n; \) the outputs vector of DMU, \( k = 1, 2, \ldots, n; \)
\( y_0 \in \mathbb{R}^n; \) the outputs vector of the target DMU;
\( u \in \mathbb{R}^n; \) the vector of input weights;
\( v \in \mathbb{R}^n; \) the vector of output weights.

In this model, the efficiency of entity evaluated is obtained as a ratio of the weighted output to the weighted input subject to the condition that the ratio for every entity is not larger than 1. Mathematically, it is described as follows:

\[
\max_{u, v} \frac{\mathbf{v}^T \mathbf{y}}{\mathbf{u}^T \mathbf{x}}
\]
subject to:
\[
\mathbf{v}^T \mathbf{y} \leq \mathbf{u}^T \mathbf{x}, \quad j = 1, 2, \ldots, n
\]
\[
\mathbf{u} \geq 0
\]
\[
\mathbf{v} \geq 0.
\]

**Definition 1 (Efficiency).** DMU is efficient if \( \theta^* = 1 \), where \( \theta^* \) is the optimal value of (9).

4. Acquisition method of uncertainty distribution

In uncertain DEA models, which will be introduced in Section 5, inputs and outputs are regarded as uncertain variables. A key problem is to determine their uncertainty distributions. Different from the method to determine probability distributions in probability theory, uncertainty distributions cannot be obtained by historical data since the sample size is too small (even no samples). Thus, we have to invite some domain experts to evaluate their belief degrees of each event will occur. For this purpose, Liu (2010a) proposed a questionnaire survey method for collecting expert’s experimental data and determine the uncertainty distribution.

The starting point is to invite one expert who is asked to complete a questionnaire about the value of some uncertain input or output variable \( x \) like “What is the value of input variable \( x \)”?

We first ask the domain expert to choose a possible value \( x \) that the uncertain demand \( x \) may take, and then quiz him “How likely is \( x \) less than or equal to \( x' \)?”

Denote the expert’s belief degree by \( x \). An expert’s experimental data \( (x, x') \) thus acquired from the domain expert.

Repeating the above process, we can obtain the following expert’s experimental data:

\[
(x_1, x_1) \quad \ldots \quad (x_n, x_n)
\]

that meet the following consistence condition (perhaps after a rearrangement)

\[
x_1 < x_2 \ldots < x_n, \quad 0 \leq x_1 < x_2 \leq \ldots \leq x_n \leq 1.
\]

Based on those expert’s experimental data, Liu (2010a) suggested an empirical uncertainty distribution,

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq x_1 \\
\frac{(x_n - x)(x - x_n)}{x_n - x_1}, & \text{if } x_1 \leq x \leq x_{i+1}, \quad 1 \leq i < n \\
1, & \text{if } x > x_n.
\end{cases}
\]

denoted by \( E(x_1, x_2, \ldots, x_n, x_n) \). Essentially, it is a type of linear interpolation method.

Assume there are \( m \) domain experts and each produces an uncertainty distribution. Then we may get \( m \) uncertainty distributions \( \Phi_1(x), \Phi_2(x), \ldots, \Phi_m(x) \). The Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. Wang, Gao, and Guo (2012) recast the Delphi method as a process to determine the uncertainty distribution. The main steps are as follows:

**Step 1:** The \( m \) domain experts provide their expert’s experimental data,

\[
(x_{ij}, y_{ij}), \quad j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m.
\]

**Step 2:** Use the \( i \)-th expert’s experimental data \( (x_{i1}, x_{i2}), \ldots, (x_{in}, y_{in}) \) to generate the \( i \)-th expert's uncertainty distribution \( \Phi_i \).

**Step 3:** Compute \( \Phi_i(x) = w_1 \Phi_1(x) + w_2 \Phi_2(x) + \ldots + w_m \Phi_m(x) \) where \( w_1, w_2, \ldots, w_m \) are convex combination coefficients.

**Step 4:** If \( |x_j - \Phi(x_j)| \) are less than a given level \( \varepsilon > 0 \), then go to Step 5. Otherwise, the \( i \)-th expert receives the summary (\( \Phi \) and reasons), and then provides a set of revised expert’s experimental data. Go to Step 2.

**Step 5:** The last \( \Phi \) is the uncertainty distribution of the customer’s demand.

5. Uncertain DEA ranking criteria

This section will give some researches to empirical uncertain DEA based on uncertainty theory introduced in Section 2. The new symbols and notations are given as follows:

\[\bar{x}_k = (\bar{x}_{k1}, \bar{x}_{k2}, \ldots, \bar{x}_{kp})\]: the uncertain input vectors of DMU, \( k = 1, 2, \ldots, n; \)
\[\bar{y}_k = (\bar{y}_{k1}, \bar{y}_{k2}, \ldots, \bar{y}_{kp})\]: the uncertain output vectors of DMU, \( k = 1, 2, \ldots, n; \)
\[\Phi_k(x) = (\Phi_{k1}(x), \Phi_{k2}(x), \ldots, \Phi_{kp}(x))\]: the uncertainty distribution vector of \( x_k = (\bar{x}_{k1}, \bar{x}_{k2}, \ldots, \bar{x}_{kp}) \), \( k = 1, 2, \ldots, n; \)
\[\Psi_k(x) = (\Psi_{k1}(x), \Psi_{k2}(x), \ldots, \Psi_{kp}(x))\]: the uncertainty distribution vector of \( y_k = (\bar{y}_{k1}, \bar{y}_{k2}, \ldots, \bar{y}_{kp}) \), \( k = 1, 2, \ldots, n, \)

In the following sections, three types of uncertain DEA fully ranking criteria are to be investigated.

5.1. The expected ranking criterion

Liu (2007, 2012) proposed the expected value operator of uncertain variable and uncertain expected value model. The essential idea of the uncertain expected DEA model is to optimize the expected value of \( \frac{\bar{y}_k}{\bar{x}_k} \); subject to some chance constraints, then we have the first type of the uncertain DEA model:

\[
\left\{ \begin{array}{l}
\max \mathbb{E} \left[ \frac{\mathbf{v}^T \mathbf{y}}{\mathbf{u}^T \mathbf{x}} \right] \\
\text{subject to :}
\end{array} \right.
\]
\[
\begin{array}{ll}
\mathcal{M}(\mathbf{v}^T \mathbf{y} \leq \mathbf{u}^T \mathbf{x}) & \geq \alpha, \quad k = 1, 2, \ldots, n \\
\mathbf{u} & \geq 0 \\
\mathbf{v} & \geq 0
\end{array}
\]
Definition 2. A vector \( (\mathbf{u}, \mathbf{v}) \geq 0 \) is called a feasible solution to the uncertain programming model \((14)\) if
\[
\mathcal{M}(\mathbf{v}^T \tilde{\mathbf{y}}_k \leq \mathbf{u}^T \tilde{\mathbf{x}}_k) \geq \alpha
\]
for \( k = 1, 2, \ldots, n \).

Definition 3. A feasible solution \( (\mathbf{u}', \mathbf{v}') \) is called an expected optimal solution to the uncertain programming model \((14)\) if
\[
\mathbb{E}
\begin{bmatrix}
\mathbf{v}^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}^T \tilde{\mathbf{x}}_0
\end{bmatrix} \geq 
\mathbb{E}
\begin{bmatrix}
\mathbf{v}'^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}'^T \tilde{\mathbf{x}}_0
\end{bmatrix}
\]
for any feasible solution \( (\mathbf{u}, \mathbf{v}) \).

Expected Ranking Criterion: The greater the optimal objective value is, the more efficient DMU0 is ranked.

Theorem 3. Assume that \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \) are independent uncertain inputs with uncertainty distribution \( \Phi_1, \Phi_2, \ldots, \Phi_n \) for each \( i \), \( i = 1, 2, \ldots, p \) and \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m \) are independent uncertain outputs with uncertainty distribution \( \Psi_1, \Psi_2, \ldots, \Psi_m \) for each \( j \), \( j = 1, 2, \ldots, q \). Then the uncertain programming model \((14)\) is equivalent to the following model:
\[
\max \frac{\mathbf{v}'^T \Psi^{-1}_k(x)}{\mathbf{u}'^T \Phi^{-1}_k(1 - \alpha)} \geq 
\max \frac{\mathbf{v}^T \Psi^{-1}_k(x)}{\mathbf{u}^T \Phi^{-1}_k(1 - \alpha)}, \ k = 1, 2, \ldots, n
\]
subject to:
\[
\begin{align*}
\mathbf{u} & \geq 0 \\
\mathbf{v} & \geq 0.
\end{align*}
\]

Proof. Since the function \( \frac{\mathbf{v}'^T \Psi^{-1}_k(x)}{\mathbf{u}'^T \Phi^{-1}_k(1 - \alpha)} \) is strictly increasing with respect to \( \tilde{\mathbf{y}}_k \) and strictly decreasing with respect to \( \tilde{\mathbf{x}}_k \) it follows from Theorem 1 that the inverse uncertainty distribution of \( \frac{\mathbf{v}'^T \Psi^{-1}_k(x)}{\mathbf{u}'^T \Phi^{-1}_k(1 - \alpha)} \) is \( \frac{\mathbf{v}^T \Psi^{-1}_k(x)}{\mathbf{u}^T \Phi^{-1}_k(1 - \alpha)} \). Thus
\[
\mathcal{M}(\mathbf{v}'^T \tilde{\mathbf{y}}_k \leq \mathbf{u}'^T \tilde{\mathbf{x}}_k) \geq \alpha \text{ holds if and only if } \mathbf{v}'^T \Psi^{-1}_k(x) \leq \mathbf{u}'^T \Phi^{-1}_k(1 - \alpha)
\]
for \( k = 1, 2, \ldots, n \). By using Theorem 2, we obtain
\[
\mathbb{E}
\begin{bmatrix}
\mathbf{v}'^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}'^T \tilde{\mathbf{x}}_0
\end{bmatrix} = \max \mathbb{E}
\begin{bmatrix}
\mathbf{v}^T \Psi^{-1}_k(x) \\
\mathbf{u}^T \Phi^{-1}_k(1 - \alpha)
\end{bmatrix}.
\]

Theorem 4. Assume that \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \) are independent uncertain inputs with uncertainty distribution \( \Phi_1, \Phi_2, \ldots, \Phi_n \) for each \( i \), \( i = 1, 2, \ldots, p \) and \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m \) are independent uncertain outputs with uncertainty distribution \( \Psi_1, \Psi_2, \ldots, \Psi_m \) for each \( j \), \( j = 1, 2, \ldots, q \). Then the uncertain programming model \((19)\) is equivalent to the following model:
\[
\mathbb{E}
\begin{bmatrix}
\mathbf{v}'^T \Psi^{-1}_k(x) \\
\mathbf{u}'^T \Phi^{-1}_k(1 - \alpha)
\end{bmatrix} \geq 
\mathbb{E}
\begin{bmatrix}
\mathbf{v}^T \Psi^{-1}_k(x) \\
\mathbf{u}^T \Phi^{-1}_k(1 - \alpha)
\end{bmatrix}, \ k = 1, 2, \ldots, n
\]
subject to:
\[
\begin{align*}
\mathbf{u} & \geq 0 \\
\mathbf{v} & \geq 0.
\end{align*}
\]

Proof. By using Theorem 1, the theorem can be easily obtained. □

5.2. The optimistic ranking criterion

Chance-constrained programming (CCP), which was initialized by Charnes and Cooper (1961), offers a powerful means for modeling stochastic decision systems. The essential idea of chance-constrained programming is to optimize some critical value with a given confidence level subject to some chance constraints. Inspired by this idea, Liu (2010a) extended it to uncertain programming models. Assuming that the decision makers want to maximize the optimistic value of the uncertain objective at given confidence level, we have the second type of DEA model:
\[
\begin{align*}
\max & \mathbb{E}
\begin{bmatrix}
\mathbf{v}'^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}'^T \tilde{\mathbf{x}}_0
\end{bmatrix} \\
\text{subject to:} & \\
\mathcal{M}(\mathbf{v}'^T \tilde{\mathbf{y}}_k \leq \mathbf{u}'^T \tilde{\mathbf{x}}_k) & \geq \alpha, \ k = 1, 2, \ldots, n \\
\mathbf{u} & \geq 0 \\
\mathbf{v} & \geq 0,
\end{align*}
\]
in which \( \alpha \in (0.5, 1) \).

Definition 4. A feasible solution \( (\mathbf{u}', \mathbf{v}') \) is called an optimistic optimal solution to the uncertain programming model \((19)\) if
\[
\mathbb{E}
\begin{bmatrix}
\mathbf{v}'^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}'^T \tilde{\mathbf{x}}_0
\end{bmatrix} \geq 
\mathbb{E}
\begin{bmatrix}
\mathbf{v}^T \tilde{\mathbf{y}}_0 \\
\mathbf{u}^T \tilde{\mathbf{x}}_0
\end{bmatrix}
\]
for any feasible solution \( (\mathbf{u}, \mathbf{v}) \).

Optimistic Ranking Criterion: The greater the optimal objective value is, the more efficient DMU0 is ranked.

Theorem 5. Assume that \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \) are independent uncertain inputs with uncertainty distribution \( \Phi_1, \Phi_2, \ldots, \Phi_n \) for each \( i \), \( i = 1, 2, \ldots, p \) and \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m \) are independent uncertain outputs with uncertainty distribution \( \Psi_1, \Psi_2, \ldots, \Psi_m \) for each \( j \), \( j = 1, 2, \ldots, q \). Then the uncertain programming model \((22)\) is equivalent to the following model:
\[ h = \max_{\mathbf{u}, \mathbf{v}} \{ \mathbf{v}^{\top} \Psi_k^{-1}(x) \leq \mathbf{u}^{\top} \Phi_k^{-1}(1 - x), \ k = 1, 2, \ldots, n \} \]

\[ \mathbf{u} \geq 0 \]

\[ \mathbf{v} \geq 0. \]

**Proof.** By using Theorem 1, the theorem can be easily obtained. \(\square\)

6. A numerical example

This example aims to illustrate the three uncertain DEA models and their corresponding ranking methods. For simplicity, we will only consider five DMUs with two inputs and two outputs which are all zigzag uncertain variables denoted by \( \mathcal{Z}(a, b, c) \). Table 1 gives the information of the DMUs.

From Tables 2–4, we can get the following conclusions:

(i) Roughly speaking, the ranking results are DMU2, DMU4, DMU5, DMU3, DMU1.

(ii) As shown in Table 5, the confidence level \( \alpha \) affects the ranking results. When \( \alpha = 0.90 \), the DMUs are ranked: DMU3, DMU4, DMU5, DMU2, DMU1. At other \( \alpha \), the DMUs are ranked: DMU2, DMU4, DMU5, DMU3, DMU1; This phenomena indicates that the ranking method in uncertain environment is more complex than the traditional ranking methods because of the inherent uncertainty contained in inputs and outputs.

(iii) Although the ranking results with different ranking criterion are uniform in this example, the three ranking criterion are different in nature.

The results of developed ranking criteria are then compared with those that are obtained from a fuzzy DEA model introduced by Guo and Tanaka (2001). The fuzzy DEA model only gives whether the DMUs are efficient under different possibility levels \( h \), as shown in Table 6. Here we only select the results of the max-
ial ranking criteria, and give the efficiency of these DMUs for comparison (see Table 7). We can find from the two tables that most efficiency properties of DMUs in our developed method are consistent with those derived from the fuzzy DEA model. However, the differences between two results also indicate our preferences of evaluating DMUs are not exactly the same under different theoretical basis.

7. Conclusion

Due to its widely practical used background, data envelopment analysis (DEA) has become a popular area of research. Since the data cannot be precisely measured in some practical cases, many papers have been published when the inputs and outputs are uncertain. This paper gave some researches to uncertain DEA based on uncertainty measure. Three uncertain DEA models have been proposed, which led to three fully ranking criteria. In order to simplify the computation of the uncertain DEA model, we have presented their equivalent crisp models. The numerical example illustrated the uncertain DEA models and the ranking methods.

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References


Table 6

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Table 7

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Efficiency of DMUs in maximal ranking results.


