Option Pricing Formula for Generalized Stock Models

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Abstract

The stock model and option pricing problem are central contents in modern finance. In this paper, generalized stock model for financial market is proposed and the European option pricing formula for the generalized stock model is computed.

Keywords: finance, fuzzy process, option pricing, Liu process

1 Introduction

Brownian motion was introduced to finance by Bachelier [1]. Samuelson [17] [18] proposed the argument that geometric Brownian motion is a good model for stock prices. In the early 1970s, Black and Scholes [3] and, independently, Metron [13] used the geometric Brownian motion to determine the prices of stock options. Stochastic financial mathematics was founded based on the assumption that stock price follows geometric Brownian motion. The Black-Scholes formula has become an indispensable tool in today’s daily financial market practice.


As a different doctrine, Liu [11] presented an alternative assumption that stock price follows geometric Liu process. Moreover, a basic stock model for fuzzy financial market was also proposed by Liu [11]. We call it Liu’s stock model in order to differentiate it from Black-Scholes stock model. Qin and Li [16] presented the European options pricing formula for Liu’s stock model.

However, the time-varying interest rate, time-varying stock drift and time-varying stock diffusion factors were not considered in Liu’s stock model. In real market, time-varying factors are necessary to be considered. We build a generalized stock model in the presence of a time-varying interest rate, time-varying stock drift and time-varying stock diffusion according to the reality. Considering the option pricing problem is a fundamental problem in financial market, we investigate the European option pricing formulas for the generalized stock model.

The rest of this paper is organized as follows. Some basic concepts about Liu process are recalled and Liu’s stock model is introduced in Section 2. The generalized stock model based on the generalized geometric Liu process is proposed in Section 3. European call and put option price formulas are derived in Sections 4 and 5. Finally, some conclusions are listed.

2 Liu’s Stock Model

Definition 1 (Liu [11]) A fuzzy process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$,

(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value $e_t$ and variance $\sigma^2 t^2$ whose membership function is

$$\mu(x) = 2 \left(1 + \exp\left(\frac{\pi|x - e_t|}{\sqrt{6}\sigma t}\right)\right)^{-1}, -\infty < x < \infty.$$  

Liu process is said to be standard if $e = 0$ and $\sigma = 1$. If $C_t$ is a Liu process, then the fuzzy process $X_t = \exp(C_t)$ is called a geometric Liu process.

It was assumed that stock price follows geometric Brownian motion, and Black-Scholes stock model was then founded based on this assumption. Liu [11] presented an alternative assumption that stock price follows geometric Liu process. Liu [11] presented a basic stock model for fuzzy financial market in which the bond price $X_t$ and the stock price $Y_t$ follow

$$\begin{cases}
  dX_t = r X_t \, dt \\
  dY_t = e Y_t \, dt + \sigma Y_t \, dC_t
\end{cases} \tag{1}$$

where $r$ is the riskless interest rate, $e$ is the stock drift, $\sigma$ is the stock diffusion, and $C_t$ is a standard Liu process. It is just a fuzzy counterpart of Black-Scholes stock model [3].

### 3 Generalized Stock Model

Let $C_t$ be a standard Liu process, and let $e(t)$ and $\sigma(t)$ be functions. Then the fuzzy differential equation

$$dY_t = e(t) Y_t \, dt + \sigma(t) Y_t \, dC_t$$

has the solution

$$Y_t = Y_0 \exp\left(\int_0^t e(s) \, ds + \int_0^t \sigma(s) \, dC_s\right),$$

which is called the generalized geometric Liu process.

The stock price given by $Y_t$ has instantaneous stock drift $e(t)$ and stock diffusion $\sigma(t)$. Both the instantaneous stock drift and stock diffusion are allowed to be time-varying.

This process includes all possible models of a stock price process that is always positive, has no jumps, and is driven by a single standard Liu process. If $e$ and $\sigma$ are constant, it degenerates into Liu's stock model. Qin and Li [16] presented the European options pricing formula for Liu's stock model.

When the interest rate is time-varying denoted by $r(t)$, and the stock price is a generalized geometric Liu process, we obtain a new stock model for fuzzy financial market in which the bond price $X_t$ and the stock price $Y_t$ follow:

$$\begin{cases}
  dX_t = r(t) X_t \, dt \\
  dY_t = e(t) Y_t \, dt + \sigma(t) Y_t \, dC_t
\end{cases} \tag{2}$$

which is called as the generalized stock model.

### 4 European Call Option Pricing Formula

A European call option gives the holder the right, but not the obligation, to buy a stock at a specified time for a specified price. In this section, we consider European call option pricing problem for the generalized stock model.

Considering the generalized stock model, we assume that a European call option has strike price $K$ and expiration time $T$. Then the payoff from buying a European call option is $(Y_T - K)^+$. Considering the time value of money, the present value of this payoff is $\exp\left(-\int_0^T r(t) \, dt\right) (Y_T - K)^+$. 

Proof: By the definition of expected value of fuzzy variable, we have

\[ f^c = \exp \left( - \int_0^T r(t) \, dt \right) \mathbb{E} \left[ \left( Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) - K \right)^+ \right] \quad (3) \]

where \( K \) is the strike price at time \( T \).

In order to calculate the option price, we solve Equation (3) and give an integral form as follows:

Theorem 1 European call option price \( f^c \) for the generalized stock model is defined as

\[ f^c = Y_0 \exp \left( \int_0^T (e(t) - r(t)) \, dt \right) \int_{K \exp \left( - \int_0^T e(t) \, dt \right) / Y_0}^{+\infty} \mathcal{Cr} \left\{ \exp \left( \int_0^T \sigma(t) \, dC_t \right) \geq u \right\} \, du. \quad (4) \]

Proof: By the definition of expected value of fuzzy variable, we have

\[ f^c = \exp \left( - \int_0^T r(t) \, dt \right) \mathbb{E} \left[ \left( Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) - K \right)^+ \right] \]

\[ = \exp \left( - \int_0^T r(t) \, dt \right) \int_0^{+\infty} \mathcal{Cr} \left\{ Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) - K \geq r \right\} \, dr \]

\[ = \exp \left( - \int_0^T r(t) \, dt \right) \int_0^{+\infty} \mathcal{Cr} \left\{ Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) - K \geq r \right\} \, dr \]

\[ = Y_0 \exp \left( \int_0^T (e(t) - r(t)) \, dt \right) \int_{K \exp \left( - \int_0^T e(t) \, dt \right) / Y_0}^{+\infty} \mathcal{Cr} \left\{ \exp \left( \int_0^T \sigma(t) \, dC_t \right) \geq u \right\} \, du. \]

5 European Put Option Pricing Formula

A European put option gives the holder the right, but not the obligation, to sell a stock at a specified time for a specified price. In this section, we consider European put option pricing problem for the generalized stock model.

Definition 2 European put option price \( f^p \) for the generalized stock model is defined as

\[ f^p = \exp \left( - \int_0^T r(t) \, dt \right) \mathbb{E} \left[ \left( K - Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) \right)^+ \right] \quad (5) \]

where \( K \) is the strike price at time \( T \).

Theorem 2 European put option price \( f^p \) for the generalized stock model is defined as

\[ f^p = Y_0 \exp \left( \int_0^T (e(t) - r(t)) \, dt \right) \int_{-\infty}^{K \exp \left( - \int_0^T e(t) \, dt \right) / Y_0} \mathcal{Cr} \left\{ \exp \left( \int_0^T \sigma(t) \, dC_t \right) \leq u \right\} \, du. \quad (6) \]

Proof: By the definition of expected value of fuzzy variable, we have

\[ f^p = \exp \left( - \int_0^T r(t) \, dt \right) \mathbb{E} \left[ \left( K - Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) \right)^+ \right] \]

\[ = \exp \left( - \int_0^T r(t) \, dt \right) \int_0^{+\infty} \mathcal{Cr} \left\{ K - Y_0 \exp \left( \int_0^T e(t) \, dt + \int_0^T \sigma(t) \, dC_t \right) \geq r \right\} \, dr \]
\[
\begin{align*}
= & \quad \exp \left( - \int_0^T r(t)\,dt \right) \int_0^{+\infty} Cr \left\{ K - Y_0 \exp \left( \int_0^T e(t)\,dt + \int_0^T \sigma(t)\,dC_t \right) \geq r \right\} \,dr \\
= & \quad Y_0 \exp \left( \int_0^T (e(t) - r(t))\,dt \right) \int_{-\infty}^{K \exp \left( - \int_0^T e(t)\,dt / Y_0 \right)} \exp \left( \int_0^T \sigma(t)\,dC_t \right) \leq u \right\} \,du.
\end{align*}
\]

6 Conclusions

In this paper, we proposed a generalized stock model for financial market and investigated the option pricing problems based on it. European call and put option price formulas for the generalized stock model were defined and computed.

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References


