Lipschitz Continuity of Liu Process

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Abstract: Liu process is a type of fuzzy process. It is a fuzzy counterpart of Brownian motion. In this paper, the continuity property of Liu process is studied. It is proved that almost all Liu paths are Lipschitz continuous.

Keywords: Fuzzy process; Liu process; Lipschitz continuity; Reflection principle; First passage.

1 Introduction

Most of fuzzy phenomena in the world is hard to be described statically. To deal with such dynamic fuzzy phenomena, Liu [7] proposed the concept of fuzzy process in the framework of credibility theory in 2008. A fuzzy process, a differential formula and a fuzzy integral is introduced by Liu in a seminar during summer holidays of 2007. Later, the community renamed those three footstones as Liu process, Liu formula and Liu integral due to their importance and usefulness, just like Brownian motion, Ito formula and Ito integral.

As a fuzzy counterpart of Brownian motion, Liu process plays an important role in describing dynamic fuzzy phenomena. Lots of mathematical tools based on Liu process have been developed. The fuzzy differential equations driven by Liu process were studied by You [13, 14, 15]. The expected value and variance of geometric Liu process were studied by Li and Qin [4]. Meanwhile, fuzzy calculus and fuzzy differential equations were introduced to finance by Liu. Several fuzzy stock models [1, 10, 2] based on Liu process were presented. The option pricing formula for fuzzy financial market was given by Qin and Li [12] in 2008.

This paper is devoted to studying continuity property of Liu process. Brownian path is an example of continuous function that is nowhere differentiable. Whether Liu paths are differentiable is what we interest in. We prove that almost all Liu paths are Lipschitz continuous. It means that almost all Liu paths are differentiable almost everywhere and have a finite variation.

This paper is organized as follows. In Section 2, we recall some basic concepts in credibility theory and fuzzy process. In Section 3, the continuity of Liu process is studied. Finally, a brief conclusion is given.

2 Preliminary


A credibility measure Cr is a real-valued set-function on the power set \( \mathcal{P} \) of a nonempty set \( \Theta \). Each element in \( \mathcal{P} \) is called an event. \( Cr(A) \) is a number assigned to event \( A \), which indicates
the credibility that $A$ will occur. A credibility measure is normal, monotonic and self-dual: $\Cr{\Theta} = 1$, $\Cr{A} \leq \Cr{B}$ whenever $A \subset B$ and $\Cr{A} + \Cr{A^c} = 1$ for any $A \in \mathcal{P}$. And it has maximality that $\Cr{\bigcup_i A_i} = \sup_i \Cr{A_i}$ for any $\{A_i\}$ with $\sup_i \Cr{A_i} < 0.5$.

In credibility theory, fuzzy variable is defined as a function from a credibility space to the set of real numbers.

**Definition 1 (Liu [5])** Let $\Theta$ be a nonempty set, $\mathcal{P}$ the power set of $\Theta$, and $\Cr{}$ a credibility measure. Then the triplet $(\Theta, \mathcal{P}, \Cr{})$ is called a credibility space.

**Definition 2 (Liu [5])** A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}, \Cr{})$ to the set of real numbers.

In Liu [7], a fuzzy process $X(t, \theta)$ is defined as a function of two variables such that $X(t^*, \theta)$ is a fuzzy variable for each $t^*$. For simplicity, we use $X_t$ to replace $X(t, \theta)$. Liu process is introduced as follows.

**Definition 3 (Liu [7])** A fuzzy process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{t+s} - C_s$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2 t^2$ whose membership function is

$$
\mu(x) = 2 \left( 1 + \exp \left( \frac{\pi|x - et|}{\sqrt{6} \sigma t} \right) \right)^{-1}, -\infty < x < +\infty.
$$

The parameters $e$ and $\sigma$ are called the drift and diffusion coefficient, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

## 3 Continuity

In this section, we study the continuity of Liu process. The following theorem claims that Liu process is continuous.

**Theorem 1** Let $C_t$ be a Liu process. Then we have

$$
\Cr{\lim_{t \rightarrow s} C_t = C_s} = 1.
$$

**Proof:** Assume that $C_t$ is a Liu process with expected value $et$ and variance $\sigma^2 t^2$. It is obvious that $(C_t - et)/\sigma$ is a standard Liu process. Write $C^0_t = (C_t - et)/\sigma$. Let $t_n$ be a sequence such that $t_n \rightarrow s$. Then we only have to prove that

$$
\Cr{\lim_{n \rightarrow \infty} (C^0_{t_n} - C^0_s) = 0} = 1.
$$

For any given integer $j > 0$, we have

$$
\Cr{\bigcap_{l=1}^{\infty} \bigcup_{n=l}^{\infty} \{ |C^0_{t_n} - C^0_s| \geq \frac{1}{m} \}} \leq \Cr{\bigcup_{n=j}^{\infty} \{ |C^0_{t_n} - C^0_s| \geq \frac{1}{m} \}}.
$$

It is obvious that

$$
\Cr{ |C^0_{t_j} - C^0_s| \geq \frac{1}{m} } = \left( 1 + \exp \left( \frac{\pi}{\sqrt{6m(t_j - s)}} \right) \right)^{-1} < 0.5.
$$
Therefore, we have
\[ \text{Cr} \left\{ \bigcup_{n=j}^{\infty} \left\{ |C_{t_n}^0 - C_{s_n}^0| \geq \frac{1}{m} \right\} \right\} = \left( 1 + \exp\left( \frac{\pi}{\sqrt{6m} \sup_{n \geq j} (t_n - s)} \right) \right)^{-1}. \]

Since \( j \) is arbitrary, we obtain
\[ \text{Cr} \left\{ \bigcap_{t_1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ |C_{t_n}^0 - C_{s_n}^0| \geq \frac{1}{m} \right\} \right\} \leq \lim_{j \to \infty} \left( 1 + \exp\left( \frac{\pi}{\sqrt{6m} \sup_{n \geq j} (t_n - s)} \right) \right)^{-1} = 0. \]

At last, we have
\[ \text{Cr} \left\{ \lim_{n \to \infty} (C_{t_n}^0 - C_{s_n}^0) = 0 \right\} = 1 - \text{Cr} \left\{ \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ |C_{t_n}^0 - C_{s_n}^0| \geq \frac{1}{m} \right\} \right\} = 1. \]

Consequently, \( \text{Cr} \left\{ \lim C_t = C_s \right\} = 1 \) holds. The proof is completed.

The following two theorems claim that almost all Liu paths are Lipschitz continuous.

**Theorem 2** Let \( C_t \) be a standard Liu process. For any given \( \theta \in \Theta \) with \( \text{Cr}\{\theta\} > 0 \), we have
\[ \sup_{0 < s < t} \frac{|C_t(\theta) - C_s(\theta)|}{|t - s|} < \infty. \]

**Proof**: Assume that the conclusion does not hold. Then for any given \( k > 0 \), there exist \( t \) and \( s \) such that
\[ \frac{|C_t(\theta) - C_s(\theta)|}{|t - s|} > k. \]

It is obvious that \( \theta \in \{ \omega : |C_t(\omega) - C_s(\omega)| > k|t - s| \} \). We have
\[ \text{Cr}\{\theta\} \leq \text{Cr}\{|C_t - C_s| > k|t - s|\} = \left( 1 + \exp\left( \frac{\pi k}{\sqrt{6}} \right) \right)^{-1}. \]

Letting \( k \) increasingly turn to infinity, we have \( \text{Cr}\{\theta\} = 0 \), which is in contradiction with \( \text{Cr}\{\theta\} > 0 \). Therefore, the assumption is not true.

The proof is completed.

**Theorem 3** Let \( C_t \) be a Liu process. For any given \( \theta \) with \( \text{Cr}\{\theta\} > 0 \), the path \( C_t(\theta) \) is Lipschitz continuous.

**Proof**: First, we prove that the standard Liu process \( C_t \) is Lipschitz continuous. For each path \( C_t(\theta) \) with \( \text{Cr}\{\theta\} > 0 \), define \( K(\theta) \) as
\[ K(\theta) = \sup_{0 < s < t} \frac{|C_t(\theta) - C_s(\theta)|}{|t - s|}. \]

It follows from Theorem 2 that \( K(\theta) < \infty \). Then \( C_t(\theta) \) is a function with respect to \( t \) and satisfies Lipschitz condition with Lipschitz constant \( K(\theta) \),
\[ |C_t(\theta) - C_s(\theta)| < K(\theta)|t - s|. \]

Thus \( C_t(\theta) \) is Lipschitz continuous for any \( \theta \) with \( \text{Cr}\{\theta\} > 0 \).

Assume that \( E_t \) is a Liu process with expected value \( et \) and variance \( \sigma^2 t^2 \). It is obvious that \( (E_t - et)/\sigma \) is a standard Liu process. Then there exists Lipschitz constant \( K(\theta) \) for each path such that
\[ \left| \frac{E_t(\theta) - et}{\sigma} - \frac{E_s(\theta) - es}{\sigma} \right| < K(\theta)|t - s|. \]
Therefore, we obtain
\[ |E_t(\theta) - E_s(\theta)| < (\sigma K(\theta) + \epsilon)|t - s|. \]

It is clear that \( E_t \) is Lipschitz continuous. The proof is completed.

**Corollary 1** For any given \( \theta \) with \( \text{Cr}\{\theta\} > 0 \), \( C_t(\theta) \) is a function with respect to \( t \). We have:
(a) Liu path \( C_t(\theta) \) is a bounded variation function.
(b) Liu path \( C_t(\theta) \) is an absolutely continuous function.
(c) Liu path \( C_t(\theta) \) is an uniformly continuous function.

**Corollary 2** The Lipschitz constant of a Liu process \( C_t \) is a fuzzy variable \( K \) defined by
\[
K(\theta) = \begin{cases} 
\sup_{0 < s < t} \frac{|C_t(\theta) - C_s(\theta)|}{|t - s|}, & \text{if \( \text{Cr}\{\theta\} > 0 \)} \\
\infty, & \text{otherwise}.
\end{cases}
\]

If \( C_t \) is a standard Liu process, for any \( k > 0 \), we have
\[
\text{Cr}\{K > k\} \leq \left( 1 + \exp\left( \frac{\pi k}{\sqrt{6}} \right) \right)^{-1}.
\]

**Remark 1** However, for any number \( \ell \in (0, 0.5) \), there is a sample \( \theta \) with \( \text{Cr}\{\theta\} = \ell \) such that the path \( C_t(\theta) \) is not differentiable in a dense set of \( t \).

4 Conclusion

When dealing with dynamic random phenomena, Brownian motion is a powerful mathematical tool. As the fuzzy counterpart of Brownian motion, Liu process performs well in dealing with dynamic fuzzy phenomena. Liu process has Lipschitz continuity. The Lipschitz continuity is a strong property, with which we can develop more powerful mathematical tools than before.

References


