Bounds for the Laplacian spectral radius of graphs

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Communicated by B. Mohar

(Received 8 April 2007; final version received 13 August 2008)

Let $G$ be a simple graph with $n$ vertices, $m$ edges, diameter $D$ and degree sequence $d_1, d_2, \ldots, d_n$, and let $\lambda_1(G)$ be the largest Laplacian eigenvalue of $G$. Denote $\Delta = \max\{d_i; 1 \leq i \leq n\}$, $(\alpha t)_i = \sum_{j-i} d_i^\alpha$ and $(\alpha m)_i = (\alpha t)_i / d_i^\alpha$, where $\alpha$ is a real number. In this article, we first give an upper bound on $\lambda_1(G)$ for a non-regular graph involving $\Delta$ and $D$; next present two upper bounds on $\lambda_1(G)$ for a connected graph in terms of $d_i$ and $(\alpha m)_i$; at last obtain a lower bound on $\lambda_1(G)$ for a connected bipartite graph in terms of $d_i$ and $(\alpha t)_i$. Some known results are shown to be the consequences of our theorems.

**Keywords:** graph; non-regular graph; Laplacian spectral radius

**AMS Subject Classifications:** 05C50; 15A18

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with $n$ vertices and $m$ edges. Denote $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any two vertices $v_i, v_j \in V(G)$, we will use the symbol $i \sim j$ to denote that vertices $v_i$ and $v_j$ are adjacent. For $v_i \in V$, the degree of $v_i$, written by $d(v_i)$ or $d_i$, is the number of edges incident with $v_i$. Let $\delta(G) = \delta$ and $\Delta(G) = \Delta$ be the minimum degree and the maximum degree of vertices of $G$, respectively. The two degrees of $v_i$ \cite{2} is the sum of the degrees of the vertices adjacent to $v_i$ and denoted by $t_i$, and the average-degree of $v_i$ is $m_i = \frac{t_i}{d_i}$. Here we define

$$(\alpha t)_i = \sum_{i-j} d_i^\alpha \quad \text{and} \quad (\alpha m)_i = (\alpha t)_i / d_i^\alpha,$$

where $\alpha$ is a real number. Note that $d_i = (0 t)_i = (0 m)_i$, $t_i = (1 t)_i$, and $m_i = (1 m)_i$.

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin’s theorem, it follows that its eigenvalues are non-negative real numbers. The eigenvalues of an $n \times n$ matrix $M$ are denoted by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$, and we assume that $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_{n-1}(M) \geq \lambda_n(M)$, while for a graph $G$, we will use $\lambda_i(L(G)) = \lambda_i$ to denote $\lambda_i(L(G))$, $i = 1, 2, \ldots, n$. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. We will call $\lambda_i(L(G))$, $i = 1, 2, \ldots, n$, the Laplacian eigenvalues and $\lambda_1(G)$ the Laplacian spectral radius of $G$.

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The eigenvalues of the Laplacian matrix $L(G)$ can be used to obtain much information about the graph, for example, estimates for expanding property, isoperimetric number, maximum cut, independence number, genus, mean distance and diameter of a graph. In particular, estimating the bound for $\lambda_1(G)$ is of great interest, and many results have been obtained [1,3–20].

In this article, we give some bounds of Laplacian spectral radius of $G$. From this, we can improve some known results, and hence these bounds are worthy of being retained in terms of precedence (i.e. for a given set of graphs, how often does the bound yield the best value among a given set of bounds, see [3]).

Let $G$ be a connected graph with the degree diagonal matrix $D(G)$ and adjacency matrix $A(G)$. Denote $Q = Q(G) = D(G) + A(G)$. Note that $Q$ is non-negative and irreducible. The Perron–Frobenius Theorem implies that $\lambda_1(Q)$ is simple and has an eigenvector $x$ with non-negative entries which must be positive if $G$ is connected.

Now we state the following lemma that will be used in the proofs of our results.

Lemma 1.1 [19] Let $G$ be a graph. Then

$$\lambda_1(G) \leq \lambda_1(Q).$$

Moreover, if $G$ is connected, then the equality holds if and only if $G$ is a bipartite graph.

2. An upper bound for a non-regular graph

It is well known that

$$\Delta + 1 \leq \lambda_1(G) \leq 2\Delta,$$

where the right-hand equality holds if and only if $G$ is a $\Delta$-regular bipartite graph. [1,6] proved that if a connected graph $G$ is non-regular, then

$$\lambda_1(G) < 2\Delta - \frac{2}{n(2D + 1)},$$

where $D$ is the diameter of $G$.

Motivated by [16] and [4], we show the following stronger inequality for a connected non-regular graph $G$ in this section.

Theorem 2.1 Let $G$ be a simple connected non-regular graph with $n$ vertices, $m$ edges, diameter $D$ and maximum degree $\Delta$. Then

$$\lambda_1(G) < 2\Delta - \frac{2n\Delta - 4m}{n(D(2n\Delta - 4m) + 1)}.\quad (3)$$

Proof Let $X = (x_1, x_2, \ldots, x_n)^T$ be the unique unit positive eigenvector of $D + A$ with eigenvalue $\lambda_1(Q)$. Then by Lemma 1.1,

$$2\Delta - \lambda_1 \geq 2\Delta - \lambda_1(Q) = 2 \sum_{i=1}^{n} (\Delta - d_i)x_i^2 + \sum_{i,j<i} (x_i - x_j)^2.\quad (4)$$

Choose vertices $v_s$, $v_t$ so that $x_s = \max_{1 \leq i \leq n} \{x_i\}$ and $x_t = \min_{1 \leq i \leq n} \{x_i\}$. Since $G$ is non-regular, we replace $\sum_{i=1}^{n} (\Delta - d_i)x_i^2$ with $(2\Delta - 2m)x_s^2$ in (4). Thus (3) holds by an argument similar to the proof of Theorem 3.5 in [16].
Because the upper bound in (3) is monotone decreasing in $n\Delta - 2m$, it follows that inequality (3) improves the bound (2).

3. Two upper bounds for a connected graph

Throughout this section, let $G$ be a simple graph of order $n$ with degree sequence $(d_1, d_2, \ldots, d_n)$. Recall that $(a \cdot t)_i = \sum_{j \sim i} d_j^a$ and $(a \cdot m)_i = (a \cdot t)_i / d_i^a$, where $a$ is a real number. Let $\tilde{D} = \text{diag}(d_1^a, \ldots, d_n^a)$.

**THEOREM 3.1** Let $G$ be a connected graph. Then

$$\lambda_1(G) \leq \min_\alpha \max_{i \sim j} \left\{ \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4(a \cdot m)_i (a \cdot m)_j}}{2} \right\},$$  

(5)

where $a$ ranges over all real numbers.

**Proof** By Lemma 1.1, $\lambda_1(G) \leq \lambda_1(Q)$. Moreover, $\lambda_1(Q) = \lambda_1(\tilde{D}^{-1}Q\tilde{D})$. Now, the $(i,j)$th element of $\tilde{D}^{-1}Q\tilde{D}$ is

$$\begin{cases}
  d_i & \text{if } i = j, \\
  \frac{d_i^a}{d_j^a} & \text{if } i \sim j, \\
  0 & \text{otherwise.}
\end{cases}$$

Let $X = (x_1, x_2, \ldots, x_n)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_1(\tilde{D}^{-1}Q\tilde{D})$. Let $x_i = \max_{1 \leq k \leq n} \{x_k\}$, and let $x_j = \max_{i \sim w} \{x_w\}$. Since $\tilde{D}^{-1}QDX = \lambda_1(\tilde{D}^{-1}Q\tilde{D})X = \lambda_1(Q)X$, we have

$$\lambda_1(Q) - d_i x_i = \sum_{j \sim w} \frac{d_j^a}{d_i^a} x_w \leq (a \cdot m)_i x_j,$$

(6)

$$\lambda_1(Q) - d_j x_j = \sum_{j \sim w} \frac{d_j^a}{d_i^a} x_w \leq (a \cdot m)_j x_i.$$  

(7)

Hence, from (6) and (7), we get

$$\lambda_1^2(Q) - (d_i + d_j)\lambda_1(Q) + d_i d_j - (a \cdot m)_i (a \cdot m)_j \leq 0.$$  

Thus

$$\lambda_1(G) \leq \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4(a \cdot m)_i (a \cdot m)_j}}{2}. \quad \blacksquare$$

**Note 3.2** If $a = 0$, then the inequality (5) is the Anderson and Morley’s bound [1]. If $a = 1$, then the inequality (5) is the Das’ bound [5, Theorem 2.14]. Let $G$ be a graph shown in Figure 1. Then the bound (5) is 6.261 when $a = 1.25$, the Das’ bound is 6.5 and Anderson and Morley’s bound is 8. Thus (5) is better than the Das’ bound and Anderson and Morley’s bound.
From (6), we have the following result.

**Corollary 3.3** Let $G$ be a connected graph. Then

$$\lambda_1(G) \leq \min_{\alpha} \max_{1 \leq i \leq n} \left\{ d_i + (\frac{\alpha}{m})i \right\}. \quad (8)$$

**Note 3.4** If $\alpha = 1$, then the inequality (8) is the Merris' bound [15].

The following two lemmas will be used in the proof of next theorem.

**Lemma 3.5** [12] Let $B$ be the adjacency matrix of the line graph of $G$. If $\rho$ is the largest eigenvalue of $B + 2I$, where $I$ is the identity matrix, then $\lambda_1 \leq \rho$.

**Lemma 3.6** [2] Let $A$ be an $n \times n$ matrix with spectral radius $\rho(A)$ and $x$ be an $n$-tuple positive vector. Then

$$\rho(A) \leq \max \left\{ \frac{(Ax)_i}{x_i} : 1 \leq i \leq n \right\},$$

where $x_i$ is the $i$th component of $x$.

Now, we give another main result of this section.

**Theorem 3.7** Let $G$ be a connected graph. Then

$$\lambda_1 \leq \min_{\alpha} \max_{i \neq j} \left\{ \frac{d_i^\alpha(d_i + (\frac{\alpha}{m})i) + d_j^\alpha(d_j + (\frac{\alpha}{m})j)}{d_i^\alpha + d_j^\alpha} \right\}.$$

**Proof** Let $B$ be the adjacency matrix of the line graph of $G$. If $i \sim j$, $x \sim y$, then $B(v_i v_j, v_x v_y) = 1$ if $v_i v_j$ and $v_x v_y$ are adjacent, and 0, otherwise. Let $W$ be a column vector whose $ij$th component is $d_i^\alpha + d_j^\alpha$. Then the $ij$th component of $(B + 2I)W$ is

$$2(d_i^\alpha + d_j^\alpha) + \sum_{i \neq x, y} (d_x^\alpha + d_y^\alpha) = 2(d_i^\alpha + d_j^\alpha) + \sum_{i \sim j} (d_i^\alpha + d_j^\alpha) + \sum_{i \sim x, y} (d_x^\alpha + d_y^\alpha).$$

**Figure 1.** A graph $G$. 

![Graph G](image)
\[ = 2 \left( d_i^\alpha + d_j^\alpha \right) + \sum_{i \neq j} \left( d_i^\alpha + d_j^\alpha \right) - \left( d_i^\alpha + d_j^\alpha \right) + \sum_{j < x} \left( d_i^\alpha + d_j^\alpha \right) - \left( d_j^\alpha + d_i^\alpha \right) \]
\[ = d_i^{\alpha+1} + d_i^\alpha (\alpha m)_i + d_j^{\alpha+1} + d_j^\alpha (\alpha m)_j \]
\[ = d_i^\alpha (d_i + (\alpha m)_i) + d_j^\alpha (d_j + (\alpha m)_j). \]

Thus
\[
\frac{(B + 2I)W_{ij}}{W_{ij}} = \frac{d_i^\alpha (d_i + (\alpha m)_i) + d_j^\alpha (d_j + (\alpha m)_j)}{d_i^\alpha + d_j^\alpha}. 
\]

The result holds from Lemmas 3.5 and 3.6. \[\Box\]

**Note 3.8** If \( \alpha = 1, \) then our result in Theorem 3.7 is the Li and Zhang’s bound [12, Theorem 3].

In [3], Brankov et al. provided automated ways to generate two large sets of conjectured upper bounds on the largest Laplacian eigenvalue of graphs. In one set, they considered a similar form of (conjectured) upper bound, depending on the edges of \( G, \) that is:

\[ \lambda_1 \leq \max_{i \neq j} f(d_i, m_i, d_j, m_j). \]

If we replace the bounding function \( f(d_i, m_i, d_j, m_j) \) by \( f(d_i, (\alpha m)_i, d_j, (\alpha m)_j) \) or by \( f(d_i^\alpha, (\alpha m)_i, d_j^\alpha, (\alpha m)_j) \) and follow the generating steps, we should get the bounds of Theorems 3.1 and 3.7, respectively. We believe that similar results can be obtained for bounds of those types by the automated ways in [3]. And we are sure that the techniques of proving Theorems 3.1 and 3.7 are able to prove a substantial number of such bounds.

### 4. A lower bound for a bipartite graph

In this section, we give a lower bound for the largest eigenvalue of the Laplacian matrix of a bipartite graph. Recall that \( (\alpha t)_i = \sum_{i \neq j} d_j^\alpha. \)

First we state the following lemma.

**Lemma 4.1** [8] Let \( A \) be a non-negative symmetric matrix and \( X \) be a unit vector of \( \mathbb{R}^n. \) If \( \rho(G) = X^T A X, \) then \( AX = \rho(A) X. \)

**Theorem 4.2** Let \( G \) be a connected bipartite graph of order \( n \) with degree sequence \( d_1, \ldots, d_n. \) Then

\[
\lambda_1(G) \geq \max_{\alpha \in \mathbb{R}} \left\{ \frac{\sum_{i=1}^n \left( d_i^{\alpha+1} + (\alpha t)_i \right)^2}{\sum_{i=1}^n d_i^{2\alpha}} \right\}. 
\]

Moreover, the equality holds in (9) for a particular value of \( \alpha < 1 \) if and only if \( G \) is a regular graph.

**Proof** Note that \( Q = D + A \) and \( L = D - A \) have the same non-zero eigenvalues by \( G \) being a bipartite graph and \( Q \) is a non-negative irreducible positive semidefinite
symmetric matrix. Let

\[ X = (x_1, x_2, \ldots, x_n)^T \]

be the unit positive eigenvector of \( Q \) corresponding to \( \lambda_1(Q) \). Take

\[ C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^{2\alpha}} (d_1^\alpha, d_2^\alpha, \ldots, d_n^\alpha)^T}. \]

Then, by an argument similar to the proof of Theorem 3.3 in [16], we have

\[ \lambda_1(G) \geq \sqrt{\frac{\sum_{i=1}^{n} (d_i^{\alpha+1} + (\alpha t)_i)^2}{\sum_{i=1}^{n} d_i^{2\alpha}}}, \]

and the equality in (9) holds for a particular value of \( \alpha \) only if \( C \) is an eigenvector of \( \lambda_1(Q) \), which implies that for all \( 1 \leq i \leq n \),

\[ d_i^{\alpha+1} + (\alpha t)_i = \lambda_1(Q)d_i^\alpha, \quad \text{i.e.} \ d_i + (\alpha m)_i = \lambda_1. \]

Suppose \( v_i, v_j \in V(G) \) with \( d_i = \Delta \) and \( d_j = \delta \). Then for \( \alpha > 0 \), we have

\[ \lambda_1 = \delta + (\alpha m)_i \leq \delta + \delta^{1-\alpha} \Delta^{\alpha} = (\Delta^{\alpha} + \delta^\alpha)\delta^{1-\alpha}. \]

and

\[ \lambda_1 = \Delta + (\alpha m)_i \geq \Delta + \Delta^{1-\alpha} \delta^\alpha = (\Delta^{\alpha} + \delta^\alpha)\Delta^{1-\alpha}. \]

Thus, for \( \alpha > 0 \), we have

\[ (\Delta^{\alpha} + \delta^\alpha)\Delta^{1-\alpha} \leq \lambda_1 \leq (\Delta^{\alpha} + \delta^\alpha)\delta^{1-\alpha}. \]

Similarly, for \( \alpha \leq 0 \),

\[ 2\Delta = \Delta + \Delta^{1-\alpha} \leq \lambda_1 \leq \delta + \delta^{1-\alpha} \delta^\alpha = 2\delta. \]

Thus, combining these two cases, we have \( \Delta = \delta \) for \( \alpha < 1 \). So \( G \) is regular.

Conversely, if \( G \) is \( d \)-regular, then \( d_i^{\alpha+1} + (\alpha t)_i = 2d^{\alpha+1} \). It follows that

\[ \lambda_1(G) = 2d = \sqrt{\frac{\sum_{i=1}^{n} (d_i^{\alpha+1} + (\alpha t)_i)^2}{\sum_{i=1}^{n} d_i^{2\alpha}}}. \]

\[ \text{Note 4.3} \quad \text{If} \ \alpha = \frac{1}{2}, \ \text{then the inequality (9) is the Shi’s bound [16, Theorem 3.3]; if} \ \alpha = 1, \ \text{then the inequality (9) is the bound by Yu et al. [18, Theorem 9]; if} \ \alpha = 0, \ \text{then the inequality (9) is the Hong’s bound (also see Corollary 10 of [18]).} \]

\[ \text{Acknowledgements} \quad \text{Many thanks to the anonymous referee for his/her many helpful comments and suggestions, which have considerably improved the presentation of the article. H. Liu was partially supported by NNSFC (Nos. 10571105, 10671081) and Scientific Research Fund of Hubei Provincial Education Department (No. D20081005). M. Lu was partially supported by NNSFC (No. 10571105).} \]
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