Strongly quasi-Hamiltonian-connected semicomplete multipartite digraphs

Mei Lu\textsuperscript{a}, Yubao Guo\textsuperscript{b}, Michel Surmacs\textsuperscript{b,\*}

\textsuperscript{a} Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
\textsuperscript{b} Lehrstuhl C für Mathematik, RWTH Aachen University, 52062 Aachen, Germany

\textbf{A R T I C L E I N F O}

Article history:
Received 13 July 2012
Received in revised form 9 August 2013
Accepted 9 August 2013

Keywords:
Semicomplete multipartite digraph
Strongly quasi-Hamiltonian connectivity

\textbf{A B S T R A C T}

A semicomplete multipartite or semicomplete \(c\)-partite digraph \(D\) is a biorientation of a \(c\)-partite graph. A semicomplete multipartite digraph \(D\) is called strongly quasi-Hamiltonian-connected, if for any two distinct vertices \(x\) and \(y\) of \(D\), there is a path \(P\) from \(x\) to \(y\) such that \(P\) contains at least one vertex from each partite set of \(D\).

In this paper we show that every 4-strong semicomplete multipartite digraph is strongly quasi-Hamiltonian-connected. This generalizes a result due to Thomassen for tournaments.

\section{Introduction and terminology}

We use Bang-Jensen and Gutin \cite{4} for terminology and notation not defined here and only consider finite simple digraphs. Let \(D\) be a digraph. By \(V(D)\) and \(A(D)\) we denote the vertex set and the arc set of \(D\), respectively. Instead of \(xy \in A(D)\) we sometimes use the notation \(x \rightarrow y\). Let \(X, Y \subseteq V(D)\) be two disjoint subsets of \(V(D)\). Then \(X \Rightarrow Y\) and \(Y \leftarrow X\) indicate that there are no arcs from a vertex in \(X\) to one in \(Y\) in \(A(D)\). \(X \Rightarrow Y\) and \(Y \leftarrow X\) indicate that there is such an arc and \(X \rightarrow Y\) denotes that \(xy\) is an arc in \(A(D)\) for all \(x \in X\) and \(y \in Y\). Furthermore, we use \(D[X]\) to denote the subdigraph of \(D\) induced by \(X\) and \(D - X\) to denote \(D[V(D) \setminus X]\). For \(x \in V(D)\) we call \(N^+_{\text{in}}(x) := \{y \in V(D) \mid xy \in A(D)\}\) the \textit{in-neighborhood} and \(N^+_{\text{out}}(x) := \{y \in V(D) \mid yx \in A(D)\}\) the \textit{out-neighborhood} of \(x\) in \(D\).

The \textit{converse} \(D^{-1}\) of \(D\) is defined as the digraph on the same vertex set \(V(D)\), but with all arcs reversed, i.e. \(D^{-1} = (V(D), \{yx \mid xy \in A(D)\})\).

By paths (cycles, respectively) in \(D\), we mean directed paths (cycles, respectively). Let \(x, y \in V(D)\) be two vertices of \(D\). We call a path from \(x\) to \(y\) in \(D\) an \((x, y)\)-\textit{path}. For an \((x, y)\)-path \(P\), the vertices in \(V(P) \setminus \{x, y\}\) are called \textit{inner vertices} of \(P\). Two \((x, y)\)-paths are \textit{internally disjoint} if their inner vertices are pairwise distinct. For a path \(P = x_1 \ldots x_l\) we use \(x_lP_{x_1}\) to denote the unique \((x_l, x_1)\)-subpath \(x_l \ldots x_1\) of \(P\) for all \(1 \leq i \leq j \leq l\). \(xCy\) is defined analogously for a cycle \(C\) and two vertices \(x, y \in V(C)\). A path (cycle, respectively) in \(D\) is called Hamiltonian if it contains all vertices of \(D\).

We call \(D\) \textit{strong} if there is an \((x, y)\)-path in \(D\) for all \(x, y \in V(D)\). A \textit{strong component} of \(D\) is a maximal induced strong subdigraph of \(D\). \(D\) is \(k\)-\textit{strong} if there are \(k\) internally disjoint \((x, y)\)-paths in \(D\) for all \(x, y \in V(D)\), which is equivalent to \(|V(D)| \geq k + 1\) and \(D - X\) is strong for all subsets \(X \subseteq V(D)\) with \(|X| < k\), by Menger’s theorem. \(D\) is called connected if its underlying graph (if we disregard the direction of the arcs in \(A(D)\)) is connected.

An \textit{acyclic digraph} is a digraph without any cycles. We call a subdigraph \(D' \subseteq D\) \textit{maximal acyclic} if \(D'\) is acyclic and every digraph \(D''\) with \(V(D'') = V(D')\) and \(D' \subseteq D'' \subseteq D\) contains a cycle.

\* Corresponding author.

\textit{E-mail addresses:} mlu@math.tsinghua.edu.cn (M. Lu), guo@mathc.rwth-aachen.de (Y. Guo), michel.surmacs@rwth-aachen.de (M. Surmacs).

\textcopyright 2013 Elsevier B.V. All rights reserved.

\url{http://dx.doi.org/10.1016/j.disc.2013.08.015}
A semicomplete multipartite or semicomplete c-partite digraph $D$ is a tuple $D = (V, A)$ such that the following holds:

- The vertex set $V$ of $D$ is the union of pairwise disjoint non-empty subsets $V_1, V_2, \ldots, V_c$, called partite sets of $D$.
- The arc set $A$ of $D$ satisfies $A \subseteq \bigcup_{i \neq j \in \{1, \ldots, c\}} (V_i \times V_j)$.
- For all $i \neq j \in \{1, \ldots, c\}$, $x \in V_i$ and $y \in V_j$, we have $xy \in A$ or $yx \in A$.

A semicomplete multipartite digraph without cycles of length 2 is called a multipartite tournament. A semicomplete digraph (tournament, respectively) is a semicomplete c-partite digraph (c-partite tournament, respectively) on $c$ vertices.

If $D$ is a semicomplete $c$-partite digraph with partite sets $V_1, \ldots, V_c$, we define the function $p : V(D) \to \{1, \ldots, c\}$, $v \mapsto p(v) :⇔ v \in V_{p(v)}$, which assigns each vertex of $D$ the index of its partite set. For a subdigraph $D' \subseteq D$ we use $V_{p(D')} := \bigcup_{v \in V(D')} V_{p(v)}$ to denote the union of partite sets of $D$ which contain a vertex of $D'$.

Because of their strong structure, tournaments have been studied extensively for the past fifty years, as evidenced by several surveys (e.g. [2,5,10,12]) on the topic, published throughout the past decades. In recent years, there has been an increasing interest to generalize tournaments to obtain larger classes of digraphs, such as locally semicomplete digraphs [1], hypertournaments [9] or multipartite tournaments [15], in which most of the important results for tournaments still hold (see [3] for more). But we often have to adjust our results to the new properties of the larger classes. If we consider multipartite tournaments, for example, Bondy showed in [6] that for all $3 \leq c < n$, there is a strong c-partite tournament on $n$ vertices that does not contain cycles of length $l > c$ and thus specifically no Hamiltonian cycle. Therefore, two of the most central results for tournaments due to Camion and Moon, respectively, cannot be extended to multipartite tournaments in their following notation.

**Theorem 1.1 ([7]).** Every strong tournament contains a Hamiltonian cycle.

**Theorem 1.2 ([11]).** Every vertex of a strong tournament on $n$ vertices is contained in a cycle of length $m$ for each $m \in \{3, 4, \ldots, n\}$.

In 1991, Goddard and Oellermann showed the following generalization of Theorem 1.2 for multipartite tournaments.

**Theorem 1.3 ([8]).** Every vertex of a strong $c$-partite tournament $D$ belongs to a cycle that contains vertices from exactly $m$ partite sets for each $m \in \{3, 4, \ldots, c\}$.

This theorem inspired a generalized concept of length. We say a path or cycle in a multipartite semicomplete digraph is of quasi-length $l$, if it contains vertices from exactly $l$ partite sets. When considering semicomplete digraphs length and quasi-length are interchangeable. Hopefully, by this new perspective, there are a multitude of results for tournaments which can be generalized in the same manner as Theorem 1.3. In this paper we will show one of them. Since Theorem 1.3 obviously also holds for semicomplete multipartite digraphs, as do many results involving tournaments for semicomplete digraphs, we will give the following definitions for the larger class.

Let $D$ be a semicomplete multipartite digraph. A path (cycle, respectively) is called a quasi-Hamiltonian path (cycle, respectively) in $D$ if it contains at least one vertex from each partite set of $D$. Two distinct vertices $x, y \in V(D)$ are strongly quasi-Hamiltonian-connected in $D$ if there is a quasi-Hamiltonian $(x, y)$-path and a quasi-Hamiltonian $(y, x)$-path in $D$. $D$ is called strongly quasi-Hamiltonian-connected if all pairs of distinct vertices of $D$ are strongly quasi-Hamiltonian-connected. Note that the definition of quasi-Hamiltonian paths, cycles and connectivity equals the standard definition of Hamiltonian paths, cycles and connectivity when we consider tournaments. With these additional notations we can give our main result.

**Theorem 1.4.** Every 4-strong semicomplete multipartite digraph is strongly quasi-Hamiltonian-connected.

This generalizes the following result for tournaments due to Thomassen.

**Corollary 1.5 ([14]).** Every 4-strong tournament is strongly Hamiltonian-connected.

Since Thomassen proved the existence of an infinite number of 3-strong tournaments which are not strongly Hamiltonian-connected in [14], Theorem 1.4 is in a sense best possible.

2. Preliminaries

Before we prove our main result, we will give some lemmata involving semicomplete multipartite digraphs. (Note that Lemmas 2.5 and 2.6 are generalized versions of lemmata used by Thomassen in [14] to show Corollary 1.5.) We begin with an obvious corollary to Theorem 1.3 which extends Theorem 1.1 to semicomplete multipartite digraphs. For convenience, we will consider a vertex to be a quasi-Hamiltonian cycle of length 0 in a non-connected semicomplete multipartite digraph, since such a digraph consists only of isolated vertices.

**Corollary 2.1.** Every vertex of a non-connected or strong semicomplete multipartite digraph $D$ is contained in a quasi-Hamiltonian cycle in $D$. 
**Proof.** The only case not covered by Theorem 1.3 is that $D$ is bipartite, i.e. $c = 2$. Let $x \in V(D)$ and $y \in N^+_{xy}(x)$ be arbitrarily chosen. Since $D$ is strong, there is an $(y, x)$-path $P$ in $D$ and thus, $x$ is contained in the quasi-Hamiltonian-cycle $C = xyPx$ in $D$. \[\square\]

In 1999, M. Tewes and L. Volkmann introduced the following useful decomposition.

**Lemma 2.2** ([13]). Let $D$ be a connected non-strong $c$-partite tournament with partite sets $V_1, \ldots, V_c$. Then there exists a unique decomposition of $V(D)$ into pairwise disjoint subsets $X_1, \ldots, X_c$, where $X_i$ is the vertex set of a strong component of $D$ or $X_i \subseteq V_k$ for some $1 \leq k \leq c$ such that $X_i \Rightarrow x_j$ for $1 \leq i < j \leq c$ and there are $x_i \in X_i$ and $y_j \in X_{j+1}$ such that $x_i \rightarrow y_j$ for $1 \leq i < r$.

Obviously, Lemma 2.2 also holds for semicomplete $c$-partite digraphs (including strong and non-connected ones) and we call the unique decomposition of $V(D)$ described therein the multipartite decomposition of $D$. Furthermore, we define the function

$$cn : V(D) \rightarrow \{1, \ldots, r\}, v \mapsto cn(v) : \Leftrightarrow v \in X_{cn(v)},$$

which assigns each vertex of $D$ the index of its decomposition set.

**Lemma 2.3.** Let $D$ be a connected non-strong semicomplete $c$-partite digraph and $X_1, \ldots, X_c$ the multipartite decomposition of $D$. Then for all $1 \leq i < j \leq r$, there is a quasi-Hamiltonian $(x_i, x_j)$-path in $D \left[\bigcup_{i \leq j \leq c} X_i\right]$ for any $x_i \in X_i$ and any $x_j \in X_j$.

**Proof.** We will show the result by induction on $j - i$. If $j - i = 1$, then there are quasi-Hamiltonian cycles $C_1 = y_1 \ldots y_1 y_1 y_1$ in $D[X_i]$, and $C_2 = z_1 \ldots z_1 z_1$ in $D[X_j]$, where $y_1 = x_i = z_1 = x_j$, by Corollary 2.1. If $y_1 \rightarrow z_1$ in $D$, then $D := y_1 \ldots y_1 z_1 \ldots z_2$ is a quasi-Hamiltonian $(x_i, x_j)$-path in $D[X_i \cup X_j]$.

Suppose that $y_1 \not\rightarrow z_1$. By the definition of multipartite digraphs, $y_1$ and $z_1$ belong to the same partite set of $D$. If $\max\{l_0, l_2\} = 1$, then both $C_1$ and $C_2$ each contain only one vertex, which means that $X_i$ and $X_j$ consist only of isolated vertices. Thus, all vertices in $X_i \cup X_j$ belong to the same partite set and therefore, we have $X_i \Rightarrow X_j$ for $j = i + 1$, a contradiction to Lemma 2.2. Hence, one of the paths $P_1 := y_1 \ldots y_1 y_1$ and $P_2 := z_1 \ldots z_2$ has length at least 1. Without loss of generality, we may assume that $l_1 \geq 2$, otherwise we consider $D^{-1}$. Since $y_{i-1} \not\rightarrow x_i = x_1$, and $X_i \Rightarrow X_i = X_i$, $P := y_1 \ldots y_{i-1} z_1 \ldots z_2$ is a quasi-Hamiltonian $(x_i, x_j)$-path in $D[X_i \cup X_j]$. This completes the proof of the base case.

Now let $j - i \geq 2$. There is a quasi-Hamiltonian $(x_i, x_j)$-path $Q = y_1 \ldots y_1$ in $D \left[\bigcup_{i \leq j \leq c} X_i\right]$ for some $y_1 \in X_{i-1}$, by induction hypothesis, and there is a quasi-Hamiltonian cycle $C = z_1 \ldots z_1 z_1$ in $D[X_j]$, where $z_1 = x_j$, by Corollary 2.1. We proceed as in the base case. If $y_1 \rightarrow z_1$ in $D$, then $P := y_1 \ldots y_1 z_1 \ldots z_2$ is a quasi-Hamiltonian $(x_i, x_j)$-path in $D \left[\bigcup_{i \leq j \leq c} X_i\right]$.

Suppose that $y_1 \not\rightarrow z_1$. By definition of multipartite digraphs, $y_1$ and $z_1$ belong to the same partite set of $D$. Again, without loss of generality, we may assume that $l_1 \geq 2$. Since $y_{i-1} \not\rightarrow x_i = x_1$, and $X_i \Rightarrow X_i = X_i$ for all $1 \leq i < j$, $P := y_1 \ldots y_{i-1} z_1 \ldots z_2$ is a quasi-Hamiltonian $(x_i, x_j)$-path in $D \left[\bigcup_{i \leq j \leq c} X_i\right]$. \[\square\]

**Lemma 2.4.** Let $D$ be a strong semicomplete $c$-partite digraph on $n \geq 3$ vertices and $x_1, x_2 \in V(D)$ two distinct vertices such that $D - x_1$ and $D - x_2$ are strong, but $D - \{x_1, x_2\}$ is not strong. Then $x_1$ and $x_2$ are strongly quasi-Hamiltonian-connected in $D$.

**Proof.** We consider the multipartite decomposition $X_1, \ldots, X_d, D - \{x_1, x_2\}$. If $r = 1$, then all vertices of $D - \{x_1, x_2\}$ belong to the same partite set of $D$. Let $y \in V(D - \{x_1, x_2\})$. Since $D - x_i$ is strong for all $i \in \{1, 2\}$, we have $x_i \rightarrow y \rightarrow x_{i-1}$ in $D$ and thus $P_i := x_ixy_{i-1}$ is a quasi-Hamiltonian $(x_i, x_{i-1})$-path in $D$ for all $i \in \{1, 2\}$. Suppose now that $r \geq 2$. Since $D - x_{i-1}$ is strong, there exist vertices $x_i \in X_i$ and $z_i \in X_i$ such that $x_i \rightarrow y_i$ and $y_i \rightarrow x_{i-1}$ in $D$ for all $i \in \{1, 2\}$. Let $\delta_0 \in \{1, 2\}$. By Lemma 2.3, there is a quasi-Hamiltonian $(y_0, z_0 \rightarrow \delta_0)$-path $Q_0$ in $D - \{x_1, x_2\}$ and therefore a quasi-Hamiltonian $(x_0, x_3 \rightarrow \delta_0)$-path $P_0 := x_0Q_0x_{3-\delta_0}$ in $D$. \[\square\]

**Lemma 2.5.** Let $D$ be a semicomplete $c$-partite digraph, $V_1, \ldots, V_c$ its partite sets and $D'$ a maximal acyclic spanning subdigraph of $D$. Then there is a path $P$ in $D'$ such that $V(P) \cap V_i \neq \emptyset$ for all $i \in \{1, \ldots, c\}$.

**Proof.** For every maximal acyclic subdigraph $H$ of $D$ there exists a vertex $x_+ \in V_+(H) := \{x \in V(D) \mid N^+_H(x) = 0\}$. Since $H$ is maximal acyclic, we have $x_+ \in A(D)$ for all $x_+ \in A(D[V(H)])$. Thus, all vertices in $V_+(H)$ belong to the same partite set of $D$. We define maximal acyclic subdigraphs $D'_i$ of $D - \{x_1, \ldots, x_{i-1}\}$ and vertices $x_i \in V(D'_i)$ recursively through $D'_1 := D'_1 - x_i \rightarrow x_{i-1}, x_i \in V_+(D'_i)$ for all $i \in \{2, \ldots, |V(D)|\}$. Let $l_i := \{i \mid 1 \leq i \leq |V(D)| - 1, V(P_{x_i}) \neq V(P_{x_{i+1}})\}$.

We order the indices contained in $l_i$ by size, i.e. $l_i = \{i_1, \ldots, i_{|l_i|}\}$ and $1 \leq i_1 < \cdots < i_{|l_i|} \leq |V(D)| - 1$. Furthermore, let $i_{|l_i|+1} := |V(D)|$.

By reverse induction on $k$, we will show that $P_{x_k} := x_kx_{k+1} \ldots x_{i_{|l_i|+1}}$ is a path in $D'_k$ for all $k \in \{1, \ldots, |l_i| + 1\}$. The base case $k = |l_i| + 1$ is trivial, since $P_{x_k} := x_{i_{|l_i|+1}}$ is obviously a path of length 0 in $D'_k$. Suppose that $1 \leq k \leq |l_i|$. By induction hypothesis, $P_{x_{k+1}}$ is a path in $D'_k$. For all $i \leq k \leq i_{k+1}$ we have $V(P_{x_{i}}) \neq V(P_{x_{i+1}})$ if and only if $i = i_k$, by definition of $l_i$. Suppose that $x_{i_k+1} \in V_+(D'_k)$. Since $x_{i_k}$ is contained in $V_+(D'_k)$ by definition, (*) implies that $x_{i_k}$ and $x_{i_{k+1}}$
are not connected in $D'$, a contradiction to $V_{p(x_0)} \neq V_{p(x_{k+1})}$. Thus, $x_{ik+1} \notin V_{i}(D'_k)$ and therefore, $N_{D'_k}^{-}(x_{ik+1}) \neq \emptyset$ holds. As a direct consequence of $V_{p(x_0)} = V_{p(x_{k+1})}$ for all $i$ with $i_k < i \leq i_{k+1}$, we have $x_i, x_{k+1} \in A(D'_k)$. Therefore, $P_i$ is a path in $D'_k$. Since the set $\{x_i \mid i \in I_{ch} \cup \{|V(D)|\}\}$ contains vertices from all partite sets of $D$, by definition of $I_{ch}$, $P := P_i$ has the desired properties. □

**Lemma 2.6.** Let $D$ be a semicomplete $c$-partite digraph, $V_1, \ldots, V_c$ its partite sets and $x, y \in V(D)$ two distinct vertices. If $D$ has an acyclic subdigraph $D'$ such that there is an $(x, z)$-path and a $(z, y)$-path in $D'$ for each vertex $z \in V(D')$ and $V(D') \cap V_i \neq \emptyset$ for all $i \in \{1, \ldots, c\}$, then $D$ contains a quasi-Hamiltonian $(x, y)$-path.

**Proof.** Let $D'$ be a maximal acyclic spanning subdigraph of $D[V(D')]$ which contains $D'$. By Lemma 2.5 and the fact that $V(D') \cap V_i \neq \emptyset$ for all $i \in \{1, \ldots, c\}$, there is a longest path $P = x_1 \ldots x_{i-1} x_i$ in $D'$ such that $V(P) \cap V_i \neq \emptyset$ for all $i \in \{1, \ldots, c\}$. If $x_1, x_i = (x, y)$, we are finished. Without loss of generality, we may assume that $x_1 \neq x$. Since $D' \subseteq D''$, there is an $(x, x_1)$-path $Q$ in $D''$. From the maximality of $I$, we derive the existence of a $z \in V(P) \cap V(Q) \setminus \{x_1\}$. Therefore, $D''$ contains an $(x, z)$-path and a $(z, x_1)$-path and thus a cycle, a contradiction. □

3. Main results

**Theorem 3.1.** Let $D$ be a strong semicomplete $c$-partite digraph, $V_1, \ldots, V_c$ its partite sets and $x, y \in V(D)$ two distinct vertices such that both $D - x$ and $D - y$ are strong. If $D$ contains three pairwise internally disjoint $(x, y)$-paths $P_k = xx_k^1 \ldots x_k^l$ with $l_k \geq 2$ for all $k \in \{1, 2, 3\}$, then $D$ contains a quasi-Hamiltonian $(x, y)$-path.

**Proof of Claim 1.** Without loss of generality, we may assume $y_1 \notin V_{p(D')}$, since $V(C) \setminus V_{p(D')} \neq \emptyset$. Let $i_0 := \max\{i \mid 1 \leq i \leq l, y_1 \notin V_{p(D')})\}$. Suppose that there is a $v \in V(C) \setminus V_{p(D')}$ such that $V_{p(D')} \cup \{v\}$ and $V(C) \setminus V_{p(D')} \setminus \{y_1\}$, respectively) and a $(u_1, u_2)$-path $P$ in $D$ whose inner vertices all belong to $V(C)$ such that $P$ contains a vertex from each partite set of $D[V(C)] \setminus V_{p(D')}$. Let $i_1 := \min\{j \mid 1 < j \leq i_0, y_1 \notin V_{p(D')}\}$. By maximality of $i$, we have $V_{p(D')} \setminus \{y_1\}$ and $V(C) \setminus V_{p(D')} \setminus \{y_1\}$, respectively, therefore, $v \in V(C) \setminus V_{p(D')}$, and $V(C) \setminus V_{p(D')} \setminus \{y_1\}$, respectively, therefore, $v \notin V(C) \setminus V_{p(D')}$. Then $P := u_1 y_1 Cy_i u_2$ where $u_1 := x_k^1$, fulfills the conditions of Claim 1, by definition of $i_0$. If $i = i_0$, then $i_1 := \min\{j \mid 1 < j \leq i_0, y_1 \notin V_{p(D')\}$. By maximality of $i_1$, we have $V_{p(D')} \setminus \{y_1\}$ and $V(C) \setminus V_{p(D')} \setminus \{y_1\}$, respectively, therefore, $v \in V(C) \setminus V_{p(D')}$. Then $P := u_1 y_1 Cy_{i_0} u_2$, where $u_1 := x_k^1$, fulfills the conditions of Claim 1, by definition of $i_0$. Finally, let $x_1$ be a maximum vertex in $D$. Without loss of generality, we may assume that $x_1 \notin V_{p(D')}$. Let $i_0 := \min\{j \mid 1 < j \leq i_0, y_1 \notin V_{p(D')}\}$, then $P := u_1 y_1 Cy_{i_0} u_2$, where $u_1 := x_k^1$, fulfills the conditions of Claim 1, by definition of $i_0$. If $i = i_0$, then $i_1 := \min\{j \mid 1 < j \leq i_0, y_1 \notin V_{p(D')}\}$, then $P := u_1 y_1 Cy_{i_0} u_2$, where $u_1 := x_k^1$, fulfills the conditions of Claim 1, by definition of $i_1$. □
Case 1. $H$ is non-connected or strong. By Corollary 2.1, there is a quasi-Hamiltonian cycle $C = y_1 \ldots y_r y_1$ in $H$. By Claim 1, there are indices $k_1, k_2 \in \{1, 2, 3\}$, $k_1 \neq k_2$, vertices $u_i \in V(P_{k_i}) \setminus \{x, y\}$ for all $i \in \{1, 2\}$ and a $(u_1, u_2)$-path $P$ in $D$ whose inner vertices all belong to $V(H)$ such that $P$ contains a vertex from each partite set of $D[V(C)] - V_{p(D')}$. Therefore, $D'' := (V(D') \cup V(P), A(D') \cup A(P))$ is an acyclic subdigraph of $D$ such that $D''$ contains a vertex from each partite set of $D$ and for every vertex $z \in V(D'')$ there is an $(x, z)$-path and a $(z, y)$-path in $D''$. Lemma 2.6 guarantees the existence of a quasi-Hamiltonian $(w_{s_1}, w_{s_2})$-path $Q$ in $H$, by Lemma 2.3 and therefore,

$$D'' := (V(D') \cup V(Q), A(D') \cup \{v_k u_{k_1}, u_{k_2} v_k\} \cup A(Q))$$

is an acyclic subgraph of $D$ such that $D''$ contains a vertex from each partite set of $D$ and for every vertex $z \in V(D'')$ there is an $(x, z)$-path and a $(z, y)$-path in $D''$, a contradiction to our assumption.

Case 2. $H$ is connected but not strong. Consider the multipartite decomposition $X_1, \ldots, X_t$ of $H$.

Subcase 2.1. There are indices $k_1, k_2 \in \{1, 2, 3\}$, $k_1 \neq k_2$ and vertices $v_{k_1} \in V(P_{k_1}) \setminus \{x, y\}$, $w_{k_1} \in X_1$, $w_{k_2} \in X_2$ and $v_{k_2} \in V(P_{k_2}) \setminus \{x, y\}$ such that $v_{k_1} \rightarrow w_{k_1}$ and $w_{k_2} \rightarrow v_{k_2}$ in $D$. There is a quasi-Hamiltonian $(w_{s_1}, w_{s_2})$-path $Q$ in $H$, by Lemma 2.3 and therefore,

$$D'' := (V(D') \cup V(Q), A(D') \cup \{v_k u_{k_1}, u_{k_2} v_k\} \cup A(Q))$$

is an acyclic subgraph of $D$ such that $D''$ contains a vertex from each partite set of $D$ and for every vertex $z \in V(D'')$ there is an $(x, z)$-path and a $(z, y)$-path in $D''$, a contradiction to Lemma 2.6.

Subcase 2.2. $X_1 := \left( \bigcup_{k \in \{1, 2, 3\} \setminus \{k_0\}} V(P_k) \setminus \{x, y\} \right) \Rightarrow X_l$ in $D$ for a $k_0 \in \{1, 2, 3\}$. Without loss of generality, we may assume that $k_0 = 1$, since the order of the paths $P_1$, $P_2$ and $P_3$ is still arbitrary at this point. There are vertices $v_{1}, v_2 \in V(P_1) \setminus \{x, y\}$, $w_1 \in X_1$ and $w_2 \in X_r$ such that $v_1 \rightarrow w_1$ and $w_2 \rightarrow v_2$ in $D$, since $D - \{x, y\}$ is strong. Let

$$I_1 := \{s \mid 1 \leq s \leq r, X_1 \Rightarrow \left( \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\} \right) \Rightarrow X_{s+1} \} \quad \text{and} \quad I_2 := \{s \mid 1 \leq s \leq r, X_{s-1} \Rightarrow \left( \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\} \right) \Rightarrow X_s \}.$$

If $I_1 = \emptyset$, then we have $X_1 \Rightarrow \left( \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\} \right)$ for all $s \in \{1, \ldots, r\}$. Analogously, we have $\left( \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\} \right) \Rightarrow X_{s_0}$ for all $s \in \{1, \ldots, r\}$, if $I_2 = \emptyset$. Thus, if $I_1 = I_2 = \emptyset$, it follows that $V(H) \subseteq V_{p(D')}$, a contradiction. Hence, without loss of generality, we may assume that $S_0 := \min I_1$ is well-defined, otherwise we consider $D^{-1}$. By definition of $S_0$, there are vertices $v_3 \in \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\}$ and $w_3 \in X_{s_0+1}$ such that $v_3 \rightarrow w_3$ in $D$. Without loss of generality, we may assume that $v_3 \in V(P_2) \setminus \{x, y\}$, since the order of the paths $P_2$ and $P_3$ is still arbitrary. Let

$$S^\downarrow := \{s \mid 1 \leq s \leq S_0, X_s \setminus V_{p(D')} \neq \emptyset\}.$$

If $S^\downarrow = \emptyset$, let $Q_1 := \emptyset$ be an empty path for convenience. Otherwise, let $S^\downarrow_0 := \max S^\downarrow$. If $S^\downarrow_0 = 1$, then there is a quasi-Hamiltonian cycle $C$ in $D[X_1]$ which contains $w_1$ by Corollary 2.1, and thus fulfills condition $(**)$ in Claim 1. Therefore, there are vertices $u_i \in V(P_i) \setminus \{x, y\}$ for all $i \in \{1, 2\}$ and a $(u_1, u_2)$-path $Q_1$ in $D$ whose inner vertices all belong to $V(C) \subseteq X_1 = \bigcup_{1 \leq s \leq S_0} X_s$ such that $Q_1$ contains a vertex from each partite set of $D[X_1] - V_{p(D')} = D \left[ \bigcup_{1 \leq s \leq S_0} X_s \right] - V_{p(D')}$. If $S^\downarrow_0 > 1$, then there are vertices $w_4, w_5 \in X_{s_0}$ and $u_2 \in V(P_2) \setminus \{x, y\}$ such that $w_4 \rightarrow u_2$ in $D$, since $X_{s_0} \Rightarrow \left( \bigcup_{2 \leq k \leq 3} V(P_k) \setminus \{x, y\} \right)$ and $V(s_0) \setminus V_{p(D')} \neq \emptyset$, by definition of $S_0$ and $S_0^\downarrow$. By Lemma 2.3, there is a quasi-Hamiltonian $(w_1, w_4)$-path $Q$ in $D \left[ \bigcup_{1 \leq s \leq S_0} X_s \right]$. If we define $Q_1 := v_1 Q_2 u_2$, again, we obtain a path from a vertex in $V(P_1) \setminus \{x, y\}$ to one in $V(P_2) \setminus \{x, y\}$ whose inner vertices all belong to $\bigcup_{1 \leq s \leq S_0} X_s$ such that $Q_1$ contains a vertex from each partite set of $D \left[ \bigcup_{1 \leq s \leq S_0} X_s \right] - V_{p(D')}$. If $S_0 = r - 1$ and $X_1 \setminus V_{p(D')} \neq \emptyset$, we define an empty path $Q_2 := \emptyset$. If $S_0 = r - 1$ and $X_1 \setminus V_{p(D')} \neq \emptyset$, then there is a quasi-Hamiltonian cycle $C$ in $D[X_{s_0+1}] = D[X_1]$, which contains $w_2$ by Corollary 2.1, and thus fulfills condition $(*)$ in Claim 1. Hence, there is a path $Q_2$ in $D$ from a vertex in $V(P_2) \setminus \{x, y\}$ to one in $V(P_1) \setminus \{x, y\}$ whose inner vertices all belong to $V(C) \subseteq X_r = \bigcup_{s_0+1 \leq s \leq r} X_s$ such that $Q_2$ contains a vertex from each partite set of $D[X_r] - V_{p(D')} = D \left[ \bigcup_{s_0+1 \leq s \leq r} X_s \right] - V_{p(D')}$. If $S_0 < r - 1$, by Lemma 2.3, we may choose a quasi-Hamiltonian $(w_3, w_2)$-path $Q'$ in $D \left[ \bigcup_{s_0+1 \leq s \leq r} X_s \right]$. Then $Q_2 := v_3 Q' v_2$ is a path in $D$ from a vertex in $V(P_1) \setminus \{x, y\}$ to one in $V(P_1) \setminus \{x, y\}$ whose inner vertices all belong to $\bigcup_{s_0+1 \leq s \leq r} X_s$ such that $Q_2$ contains a vertex from each partite set of $D \left[ \bigcup_{s_0+1 \leq s \leq r} X_s \right] - V_{p(D')}$. By construction, the paths $Q_1$ and $Q_2$ have at most one vertex in common (the first one of $Q_1$ and the last one of $Q_2$) and $V(Q_1) \cup V(Q_2)$ contains a vertex from each partite set of $H - V_{p(D')}$, since $\bigcup_{s_0+1 \leq s \leq r} X_s \setminus V_{p(D')} = \emptyset$, by maximality of $S_0^\downarrow$. Therefore,

$$D'' := (V(D') \cup V(Q_1) \cup V(Q_2), A(D') \cup A(Q_1) \cup A(Q_2))$$

is an acyclic subgraph of $D$ such that $D''$ contains a vertex from each partite set of $D$ and for every vertex $z \in V(D'')$ there is an $(x, z)$-path and a $(z, y)$-path in $D''$, a contradiction to Lemma 2.6. $\square$
Our main result follows as a direct consequence.

**Theorem 3.2.** Every 4-strong semicomplete multipartite digraph is strongly quasi-Hamiltonian-connected.

**Acknowledgments**

We are grateful to the anonymous referees, whose remarks helped improve the writing as well as the comprehensibility of this paper.

**References**