On the spectral radius of graphs

Aimei Yu a,*, Mei Lu b, Feng Tian a

a Institute of Systems Science, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China
b Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Received 17 August 2003; accepted 12 January 2004

Submitted by R.A. Brualdi

Abstract

Let G be a simple undirected graph. For v ∈ V(G), the 2-degree of v is the sum of the degrees of the vertices adjacent to v. Denote by ρ(G) and µ(G) the spectral radius of the adjacency matrix and the Laplacian matrix of G, respectively. In this paper, we present two lower bounds of ρ(G) and µ(G) in terms of the degrees and the 2-degrees of vertices.

AMS classification: 05C50; 15A18

Keywords: Adjacency matrix; Laplacian matrix; Spectral radius; 2-Degree; Pseudo-regular graph; Pseudo-semiregular bipartite graph

1. Introduction

Let G = (V, E) be a simple undirected graph with n vertices and m edges. For v_i ∈ V, the degree of v_i, written by d_i, is the number of edges incident with v_i. Let δ(G) = δ and Δ(G) = Δ be the minimum degree and the maximum degree of vertices of G, respectively. A graph G is called regular if every vertex of G has equal...
degree. A bipartite graph is called semiregular if each vertex in the same part of a bipartition has the same degree.

The 2-degree of $v_i$ [2] is the sum of the degrees of the vertices adjacent to $v_i$ and denoted by $t_i$. We call $\frac{t_i}{2}$ the average-degree of $v_i$. A graph $G$ is called pseudo-regular if every vertex of $G$ has equal average-degree. A bipartite graph is called pseudo-semiregular if each vertex in the same part of a bipartition has the same average-degree. Obviously, any regular graph is a pseudo-regular graph and any semiregular bipartite graph is a pseudo-semiregular bipartite graph. Conversely, a pseudo-regular graph may be not a regular graph, such as $S(K_{1,3})$, and a pseudo-semiregular bipartite graph may be not a semiregular bipartite graph, such as $S(K_{1,n-1})$ ($n \geq 5$), where $S(K_{1,j})$ is the graph obtained by subdividing each edge of $K_{1,j}$ one time.

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Clearly, $A(G)$ and $L(G)$ are real symmetric matrices. Hence their eigenvalues are real numbers. We denote by $\rho(M)$ the largest eigenvalue of a symmetric matrix $M$. For a graph $G$, we denote by $\rho(G)$ the largest eigenvalue of $A(G)$ and call it the spectral radius of $G$; we denote by $\mu(G)$ the largest eigenvalue of $L(G)$ and call it the Laplacian spectral radius of $G$. When $G$ is connected, $A(G)$ is irreducible and so by Perron–Frobenius Theorem, $\rho(G)$ is simple.

Up to now, many bounds for $\rho(G)$ and $\mu(G)$ were given (see, for instance, [1–12]), but most of them are upper bounds. In the paper, we give two new lower bounds on $\rho(G)$ and $\mu(G)$ of $G$ in terms of the degrees and the 2-degrees of vertices of $G$, from which we can get some known results.

2. Lemmas and results

**Lemma 1** [5]. Let $A$ be a nonnegative symmetric matrix and $x$ be a unit vector of $\mathbb{R}^n$. If $\rho(A) = x^T A x$, then $A x = \rho(A) x$.

**Lemma 2** [13]. Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a simple graph. Then
\[
\sum_{i=1}^{n} d_i^2 \leq \left( \sum_{i=1}^{n} \sqrt{d_i} \right)^2,
\]
with equality if and only if the graph is empty.

**Lemma 3** [1]. Let $G$ be a simple connected graph. Then
\[
\mu(G) \leq \max\{d(u) + d(v) : uv \in E(G)\},
\]
with equality if and only if $G$ is a regular or semiregular bipartite graph.

The following theorem is one of our main results.
Theorem 4. Let $G$ be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\rho(G) \geq \sqrt{\sum_{i=1}^{n} t_i^2},$$

with equality if and only if $G$ is a pseudo-regular graph or a pseudo-semiregular bipartite graph.

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the unit positive eigenvector of $A$ corresponding to $\rho(A)$. Take

$$C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} (d_1, d_2, \ldots, d_n)^T.$$ 

Noting that $C$ is a unit positive vector, we have

$$\rho(G) = \rho(A) = \sqrt{\rho(A^2)} = \sqrt{X^T A^2 X} \geq \sqrt{C^T A^2 C}.$$

Since

$$AC = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \left( \sum_{j=1}^{n} a_{1j} d_j, \ldots, \sum_{j=1}^{n} a_{nj} d_j \right)^T$$

$$= \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} (t_1, \ldots, t_n)^T, \tag{*}$$

we have

$$\rho(G) = \rho(A) \geq \sqrt{C^T A^2 C} = \sqrt{\sum_{i=1}^{n} t_i^2}.$$ 

If the equality holds, then

$$\rho(A^2) = C^T A^2 C.$$ 

By Lemma 1, $A^2 C = \rho(A^2) C$. If the multiplicity of $\rho(A^2)$ is one, then $X = C$, which implies $t_i = \rho(G) d_i$ ($1 \leq i \leq n$). Hence $G$ is a pseudo-regular graph. Otherwise, the multiplicity of $\rho(A^2) = (\rho(A))^2$ is two, which implies that $-\rho(A)$ is also an eigenvalue of $G$. Then $G$ is a connected bipartite graph (see Theorem 3.4 in [3]). Without loss of generality, we assume

$$A = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix},$$

where $B = (b_{i,j})$ is an $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$
and
\[ C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \]
where \( X_1 = (x_1, x_2, \ldots, x_n)^T \), \( X_2 = (x_{n+1}, x_{n+2}, \ldots, x_n)^T \), \( C_1 = (d_1, d_2, \ldots, d_{n_1})^T \) and \( C_2 = (d_{n_1+1}, d_{n_1+2}, \ldots, d_n)^T \). Since
\[ A^2 = \begin{pmatrix} B B^T & 0 \\ 0 & B^T B \end{pmatrix}, \]
we have
\[ BB^T C_1 = \rho(A^2) C_1, \quad B^T BC_2 = \rho(A^2) C_2 \]
and
\[ BB^T X_1 = \rho(A^2) X_1, \quad B^T BX_2 = \rho(A^2) X_2. \]
Noting that \( BB^T \) and \( B^T B \) have the same nonzero eigenvalues, \( \rho(A^2) \) is the spectral radius of \( BB^T \) and its multiplicity is one. So \( X_1 = p_1 C_1 \) \((p_1 \) is a constant\), which implies \( t_i d_i = t_j d_j \) \((1 \leq i < j \leq n_1)\). Similarly, \( X_2 = p_2 C_2 \) \((p_2 \) is a constant\), which implies \( t_i d_i = t_j d_j \) \((n_1 + 1 \leq i < j \leq n)\). Hence \( G \) is a pseudo-semiregular graph.

Conversely, if \( G \) is pseudo-regular, then \( t_i d_i = p \) \((1 \leq i \leq n)\) is a constant, which implies \( AC = p C \). It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. Hence \( \rho(G) = p = \frac{\sqrt{t_1^2 + t_2^2 + \cdots + t_n^2}}{\sqrt{d_1^2 + d_2^2 + \cdots + d_n^2}} \).

If \( G \) is a pseudo-semiregular bipartite graph, we assume
\[ A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}, \]
\( \frac{t_i}{d_i} = p_1 \) \((1 \leq i \leq n_1)\) and \( \frac{t_i}{d_i} = p_2 \) \((n_1 + 1 \leq i \leq n)\), where \( B = (b_{i,j}) \) is an \( n_1 \times n_2 \) matrix with \( n_1 + n_2 = n \). Let \( C_1 = (d_1, d_2, \ldots, d_{n_1})^T \) and \( C_2 = (d_{n_1+1}, d_{n_1+2}, \ldots, d_n)^T \). So for each \( i \) \((1 \leq i \leq n_1)\), the \( i \)th element of \( BB^T C_1 \) is
\[ r_i(BB^T C_1) = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} b_{ik} b_{jk} d_j = \sum_{k=1}^{n_2} b_{ik} \sum_{j=1}^{n_1} b_{jk} d_j = \sum_{k=1}^{n_2} b_{ik} p_2 d_{n_1+k} = p_1 p_2 d_i. \]
Similarly, \( r_j(B^T BC_2) = p_1 p_2 d_{n_1+j} \), for each \( j \) \((1 \leq j \leq n_2)\). Hence \( A^2 C = p_1 p_2 C \), where \( C = \sqrt{\frac{1}{d_1^2 + d_2^2 + \cdots + d_n^2}} (d_1, d_2, \ldots, d_n)^T \). It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. So
\[ \rho(A^2) = p_1 p_2 = C^T A^2 C. \]

From equality (\(^*\)), we have
\[ \rho(A^2) = p_1 p_2 = \frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}. \]

It follows that
\[ \rho(G) = \sqrt{\frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}}. \]

This completes the proof. \(\square\)

**Corollary 5**

(1) Let \( G \) be a pseudo-regular graph with \( t(v) = pd(v) \) for each \( v \in V(G) \), then \( \rho(G) = p \).

(2) Let \( G \) be a pseudo-semiregular bipartite graph with the bipartition \( (X, Y) \). If \( t(v) = p_x d(v) \) for each \( v \in X \) and \( t(v) = p_y d(v) \) for each \( v \in Y \), then \( \rho(G) = \sqrt{p_x p_y} \).

According to Corollary 5, it is very easy to compute the spectral radius of pseudo-regular graphs and pseudo-semiregular bipartite graphs.

**Example.** Let \( S(K_{1,k}) \) be the graph obtained by subdividing each edge of \( K_{1,k} \) one time and \( G_1 \) and \( G_2 \) are the graphs shown in Fig. 1. Obviously, \( G_1 \) is a pseudo-regular graph and \( G_2 \) is a pseudo-semiregular bipartite graph. When \( k = 3 \), \( S(K_{1,k}) \) is a pseudo-regular graph; otherwise, \( S(K_{1,k}) \) is a pseudo-semiregular bipartite graph. Hence we have the following results:

(1) \( \rho(G_1) = 4 \).

(2) \( \rho(S(K_{1,k})) = \sqrt{k + 1} \).

(3) \( \rho(G_2) = 2\sqrt{2} \).

![Fig. 1.](image)
Corollary 6. Let $G$ be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\rho(G) \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2},$$

with equality if and only if $G$ is either a regular connected graph or a semiregular connected bipartite graph.

Proof. By Theorem 4 and the Cauchy–Schwarz inequality,

$$\rho(G) \geq \sqrt{\frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}} \geq \sqrt{\frac{(t_1 + t_2 + \cdots + t_n)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}}.$$

Since

$$t_1 + t_2 + \cdots + t_n = d_1^2 + d_2^2 + \cdots + d_n^2,$$

we have

$$\rho(G) \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}.$$

If the equality holds, $G$ is a pseudo-regular graph or a pseudo-semiregular bipartite graph (by Theorem 4) with $t_i = t_j$ for all $1 \leq i < j \leq n$. Thus $G$ is a regular connected graph or a semiregular connected graph. Conversely, if $G$ is a regular connected graph, the equality holds immediately. If $G$ is a semiregular connected bipartite graph, we assume that $d(v_1) = \cdots = d(v_{n_1}) = \Delta$ and $d(v_{n_1+1}) = \cdots = d(v_n) = \delta$. Since $n_1 \Delta = (n - n_1)\delta$, $\sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2} = \sqrt{\Delta \delta}$. By Corollary 5, we have $\rho(G) = \sqrt{\Delta \delta}$. Thus the equality holds. \(\square\)

Corollary 7. Let $G$ be a simple connected graph. Then

$$\rho(G) \geq \frac{2m}{n} \geq \delta.$$

Proof. By Corollary 6 and the Cauchy–Schwarz inequality,

$$\rho(G) \geq \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2} \geq \sqrt{\frac{(\sum_{i=1}^{n} d_i)^2}{n^2}} = \frac{2m}{n} \geq \delta. \quad \square$$

By Lemma 2 and Theorem 4, we have the following:

Corollary 8. Let $G$ be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\rho(G) \geq \frac{\sqrt{t_1^2 + t_2^2 + \cdots + t_n^2}}{\sqrt{d_1^2} + \sqrt{d_2^2} + \cdots + \sqrt{d_n^2}}.$$
Now we show another main result of the paper.

**Theorem 9.** Let $G$ be a connected bipartite graph with degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\mu(G) \geq \sqrt{\frac{\sum_{i=1}^{n} (d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}},$$

where the equality holds if and only if $G$ is a semiregular connected bipartite graph.

**Proof.** Note that $D + A$ and $D - A$ have the same nonzero eigenvalues by $G$ being a bipartite graph and $D + A$ is a nonnegative irreducible positive semidefinite symmetric matrix.

Let $X = (x_1, x_2, \ldots, x_n)^T$ be the unit positive eigenvector of $D + A$ corresponding to $\mu(G)$. Take

$$C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} (d_1, d_2, \ldots, d_n)^T.$$

Then

$$\mu(G) = \sqrt{\rho((D + A)^2)} = \sqrt{X^T (D + A)^2 X} \geq \sqrt{C^T (D + A)^2 C}.$$

Since

$$(D + A)C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \left( d_1^2 + \sum_{j=1}^{n} a_{ij}d_j, \ldots, d_n^2 + \sum_{j=1}^{n} a_{nj}d_j \right)^T$$

$$= \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \left( d_1^2 + t_1, \ldots, d_n^2 + t_n \right)^T,$$

we have

$$\mu(G) \geq \sqrt{C^T (D + A)^2 C} = \sqrt{\frac{\sum_{i=1}^{n} (d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}}.$$

If the equality holds, then

$$\rho((D + A)^2) = C^T (D + A)^2 C,$$

which implies that $(D + A)^2 C = \rho((D + A)^2) C$ (by Lemma 1). Since $D + A$ is a nonnegative irreducible positive semidefinite matrix, all eigenvalues of $D + A$ are nonnegative. By Perron–Frobenius Theorem, the multiplicity of $\rho(D + A)$ is one. Since $\rho((D + A)^2) = (\rho(D + A))^2$, we have the multiplicity of $\rho((D + A)^2)$ is one. Hence, if the equality holds then $X = C$. By $C^T (D + A)C = (D + A)C$, we have $\rho(D + A) d_i = d_i^2 + t_i$ for $i = 1, 2, \ldots, n$. Thus $d_i + t_i/d_i = d_j + t_j/d_j$ for all $i \neq j$. Assume, without loss of generality, that $d_1 = A, d_2 = B$ and $A \neq B$. Then we have
\[ A + t_1/A = \delta + t_2/\delta. \]

Since \( t_1 \geq \Delta \delta \) and \( t_2 \leq \delta \Delta \),
\[ A + \delta \leq A + t_1/A = \delta + t_2/\delta \leq A + \delta. \]

Thus we must have \( t_1 = \Delta \delta = t_2 \). This implies \( d(v) = A \) or \( d(v) = \delta \) for all \( v \in V(G) \) by \( G \) being connected and \( uv \notin E(G) \) if \( d(u) = d(v) \). Let \( Y_1 = \{v : d(v) = A\} \) and \( Y_2 = \{v : d(v) = \delta\} \). Then \( G = (Y_1, Y_2; E) \) is a semiregular connected bipartite graph.

Conversely, assume that \( G \) is a semiregular connected bipartite graph with \( d(v_1) = \cdots = d(v_{n_1}) = A \) and \( d(v_{n_1+1}) = \cdots = d(v_n) = \delta \). Note that \( n_1A = (n - n_1)\delta \). Then
\[
\sqrt{\frac{\sum_{i=1}^{n_1}(d_i^2 + t_i)^2}{\sum_{i=1}^{n_1}d_i^2}} = \sqrt{\frac{n_1(A^2 + A\delta)^2 + (n - n_1)(\delta^2 + \delta A)^2}{n_1A^2 + (n - n_1)\delta^2}} = \delta + A.
\]

By Lemma 3, \( \mu(G) = A + \delta \) and so the equality holds.  

**Corollary 10** [5]. Let \( G \) be a simple connected bipartite graph with degree sequence \( d_1, d_2, \ldots, d_n \). Then
\[
\mu(G) \geq 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2},
\]
where the equality holds if and only if \( G \) is a regular connected bipartite graph.

**Proof.** By Theorem 9 and the Cauchy–Schwarz inequality, we have
\[
\mu(G) \geq \sqrt{\frac{\sum_{i=1}^{n}(d_i^2 + t_i)^2}{\sum_{i=1}^{n}d_i^2}} \geq \sqrt{\frac{(d_1^2 + t_1 + d_2^2 + t_2 + \cdots + d_n^2 + t_n)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}} = \sqrt{\frac{(2d_1^2 + 2d_2^2 + \cdots + 2d_n^2)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}} = 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} d_i^2}.
\]

If the equality holds, \( G \) is a semiregular connected bipartite graph (by Theorem 9) with \( d_i^2 + t_i = d_j^2 + t_j \) for \( 1 \leq i < j \leq n \). Without loss of generality, assume that \( d_1 = A \) and \( d_2 = \delta \). Then we have \( A^2 + \Delta A = \delta^2 + \delta A \), which implies \( A = \delta \). Hence \( G \) is a regular connected bipartite graph. Conversely, if \( G \) is a regular connected bipartite graph, by Lemma 3, the equality holds immediately.  

Corollary 11. Let $G$ be a simple connected bipartite graph. Then

$$\mu(G) \geq \frac{4m}{n} \geq 2\delta.$$

Acknowledgements

Many thanks to the anonymous referees for their many helpful comments and suggestions, which led to an improved version of the paper.

References